# RECONSTRUCTED PHASE SPACE OF A VECTOR TIME SERIES[1]

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## ABSTRACT

*Given a one dimensional time series, attractor behavior of an underlying dynamical system can be analyzed by unfolding the time series to a higher dimensional phase space by employing Taken's theorem of embedding. Conventional assumptions include the fact that only one observable of the system is available for attractor reconstruction and characterization. In this paper, we present a technique to perform meaningful reconstruction in the phase space using multivariate data. We demonstrate the efficacy of this approach by embedding a vector time series (comprised of the three variables of a Lorentz system) and show that the Lyapunov spectra (one of the dynamical invariants) of an attractor reconstructed from such a time series is comparable to the spectra estimated using conventional embedding. We also demonstrate that this method can be used to reliably estimate the Lyapunov spectra from a system of uncorrelated state variables.*

## 1. INTRODUCTION

The attractor characteristics of a dynamical system that generates a scalar time series can be estimated by unfolding the observed time series to a higher dimensional, reconstructed phase space [1] and measuring dynamical invariants in this higher dimensional space. This property has been exploited in studies involving non-linear dynamical systems by a technique called embedding. Topologically, an embedding problem is posed as finding a one-to-one map between points on the original system attractor and the attractor in the reconstructed phase space. In other words, embedding refers to finding the optimal mapping which when applied to the observed time series will map it to a higher dimensional space, revealing information about the original attractor. Since the underlying properties of a nonlinear system are best studied by analyzing the attractor, our intuitions guide us to believe that signal classification and prediction for signals generated by a nonlinear system will be most effective in a higher dimensional reconstructed attractor space.

Nonlinear systems which exhibit sensitivity to initial conditions are called chaotic systems. For observables from these systems, linear analysis tools may fail to provide a full description of the system behavior. It is hence desired to characterize chaotic signals using tools that capture the topological information of the system's attractor. Lyapunov exponents are one such measure. Apart from being able to distinguish between fixed points, periodic, quasi-periodic and chaotic motions, these exponents also quantify the extent of chaos in an observed time series.

In this paper, we show that conventional reconstruction techniques can be extended to vector observables of the system being studied. We demonstrate this by showing that Lyapunov exponents of a system's attractor are preserved when we use more than a single state space variable for reconstruction. In fact, results indicate that embedding a vector time series can provide a more accurate representation of the underlying system behaviour. We also show that a vector time series comprised of uncorrelated variables (e.g., derived from a system described by a set of loosely coupled differential equations) can provide reliable estimates of dynamical invariants.

The paper is organized as follows. In Section 2, we explain the extension of the conventional phase space reconstruction technique to a vector time series. In Section 3, for completeness, we describe Lyapunov exponents as an invariant measure of the dynamical system. We present an outline of the algorithm employed for estimating these exponents from scalar and vector time series. In Section 4, we describe the experimental setup used in this work. In Section 5, we provide a summary and explanation of the results. We conclude the paper by discussing some potential applications of employing the concept of vector embedding.

## 2. RECONSTRUCTING THE PHASE SPACE

Computation of dynamical invariants (e.g., Lyapunov exponents) assumes knowledge of the dynamics of a system. Typically, to reconstruct these dynamics from an observed scalar time series, we project the time series onto a pseudo phase space (equivalent to the original phase space in terms of the system invariants). This pseudo phase space [1], [2], [3] is called the Reconstructed Phase Space (RPS). For projecting the time series in this space, we need to know the inherent system dimension, d. Given an estimate of the

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system dimension, an upper bound on the dimension of the RPS is provided by Taken's theorem [2]. A time series generated by a nonlinear system can typically be embedded in low-dimensional spaces.

In time delay embedding, evolution of the system's states in the original state-space is approximated by a phase space comprising of time-delayed coordinates. Equation 1 illustrates the RPS matrix, $X$, obtained from embedding a scalar time series.

$$X = \begin{pmatrix} x_0 & x_\tau & \cdots & x_{(m-1)\tau} \\ x_1 & x_{1+\tau} & \cdots & x_{1+(m-1)\tau} \\ x_2 & x_{2+\tau} & \cdots & x_{2+(m-1)\tau} \\ \vdots & & \vdots & \end{pmatrix} \quad (1)$$

Here $\tau$ is the time delay and $m$ is the embedding dimension. In a conventional nonlinear analysis, the time series being embedded is a single observable of the system. In this work, we extend the embedding technique to a system with more than one observable, resulting in a vector time series $\vec{x}$. Given a vector time series, a reconstructed phase space of the system's attractor can be created by stacking time delayed versions of vectors from the data vector stream. The corresponding embedding is defined by equation 2.

$$X = \begin{pmatrix} \vec{x}_0 & \vec{x}_\tau & \cdots & \vec{x}_{(m-1)\tau} \\ \vec{x}_1 & \vec{x}_{1+\tau} & \cdots & \vec{x}_{1+(m-1)\tau} \\ \vec{x}_2 & \vec{x}_{2+\tau} & \cdots & \vec{x}_{2+(m-1)\tau} \\ \vdots & & \vdots & \end{pmatrix} \quad (2)$$

Here, $\vec{x}_i$ represents a row vector of the i'th sample of the vector-time series. In other words, we study the dynamics in a pseudo state space comprised of time delayed vectors as coordinates. We can extend this to a more general formulation, where the embedding dimension and time delay is different for each component of the vector observable.

### 2.1. Singular Value Decomposition-based Embedding

A method of embedding that has been successfully applied for various nonlinear analyses of scalar time series is Singular Value Decomposition (SVD) based embedding [4]. It works in two stages. In the first stage, the original time series is embedded into a higher dimensional space using time delay embedding with a delay of one sample. The dimensionality of this space is referred to as the SVD window size. In the next stage the embedded matrix is reduced to a lower dimensional space by a linear transformation (the singular vectors matrix).

In this paper, we extend this method to a vector observable of the system. This is achieved by first time delay embedding the vectors to form a matrix in a higher dimensional space. The second stage is similar to that of SVD-based scalar embedding. Experimental results are provided that demonstrate the efficacy of this technique, measured by the Lyapunov exponents of the unfolded attractor.

## 3. LYAPUNOV EXPONENTS

The analysis of separation in time of two trajectories with infinitely close initial points is very important in quantifying chaos in nonlinear dynamical systems [1]. For a system whose states evolve over the function $f$, we analyze the following equation:

$$\Delta x(t) \approx \Delta x(0) \frac{d}{dx}(f^N)x(0). \quad (3)$$

To quantify the average rate of separation, we assume that the rate of growth (or decay) is exponential in time. Hence we can define a set of exponents, $\lambda_i$, as described by equation 4:

$$\lambda_i = \lim_{n\to\infty} \frac{1}{n} \ln\left( \mathrm{eig} \prod_{\mathrm{p}=0}^{\mathrm{n}} \mathrm{J(p)} \right), \; i = 1,2,...,m, \quad (4)$$

where $J$ is the Jacobian of the system as the point $p$ moves around the attractor. These set of exponents are one of the characteristic invariants of the system and are called Lyapunov exponents (LEs). There are as many LEs as the dimension of the system. For a dynamical system with a bounded attractor, the sum of all LEs should be less than or equal to zero. Zero exponents indicate that the system is a flow, while the positive ones indicate that the system is chaotic. Negative exponents characterize a system's tendency to pull an evolving trajectory towards the basin of attraction.

The algorithm employed to estimate these exponents from a time series [4], [5], [7] can be summarized in six steps. First, we embed the input time series to generate the RPS matrix. Each row of this matrix represents a point on the attractor. Second, using the first point as a center, we form a neighborhood matrix, each row of which is obtained by subtracting a neighbor from the center. Third, we find the evolution of each neighbor and form an evolved neighborhood matrix. Fourth, a trajectory matrix is obtained by multiplying the pseudo-inverse of neighborhood matrix with the evolved neighborhood matrix. Fifth, the LEs are calculated from the eigenvalues of the trajectory matrix. Finally, these exponents are averaged by evolving the center point through the trajectory. Since direct averaging has numerical problems, an iterative QR decomposition method (Treppen iteration) is preferred.

## 4. EXPERIMENTAL SETUP

To demonstrate the efficacy of the procedure for vector embedding, we consider the accuracy of Lyapunov exponents on a well known chaotic system, the Lorentz system of differential equations. The parameters for which the system was tested are $\sigma = 16.0$, $r = 40.0$, and $b = 4.0$. The dimensionality of this system is three and hence there are three LEs for this system. These can be calculated numerically from the set of differential equations [6] and are given by (+1.37, 0.0, -22.37).

We performed experiments on both SVD-based scalar and vector embedding. For scalar embedding experiments, we considered the evolution of one variable of the attractor's state space as the scalar observable of the system. For vector embedding experiments, we used all the three variables of the state space as a vector observable of the system.

The final embedding dimension of the attractor was set to three for all experiments. We studied the accuracy of Lyapunov exponent estimates as a function of various parameters, i.e., SVD window size, number of neighbors and the evolve step size.

As discussed previously, SVD window size refers to the dimension of the initial RPS matrix which is then reduced (by ignoring dimensions corresponding to small singular values) to a lower dimensional space using SVD. This technique of embedding has an additional benefit in that it reduces the effects of noise in the system's analysis. Setting a very low value for SVD window size neutralizes the noise reduction property of SVD while at very high values of this parameter we lose the high frequency information since SVD based dimensionality reduction of the RPS matrix acts like a low pass filter.

The second parameter, number of neighbors refers to the number of neighboring points used to analyze the local dynamics of the attractor's trajectory. A small number of neighbors implies that we may not have sufficient number of data points to capture the local dynamics of the attractor. On the other hand, too many neighbors may not allow for an accurate description of the local behavior.

The "evolution step" is another parameter, referring to the number of points to jump on the attractor for studying the evolution of the neighborhood. At very small values, the effects of noise may dominate the analysis while at very large values the local evolution analysis may no longer be valid.

We analyzed the performance of Lyapunov spectra estimation for scalar and vector embedding as a function of the size of the data-set. For all other experiments reported in this paper, we used 30,000 data points from the Lorentz attractor which were generated by solving the Lorentz system of differential equations using Runge-Kutta numerical integration. An integration time step of 0.001 sec was used and the sampling rate was set to 100 Hz, as described in [6].

The Lorentz system of equations is a tightly coupled system, i.e., there is a high degree of correlation between the three variables of the system. On the other hand, many practical systems that produce vector time series do not exhibit this property. As an example, consider a vector stream comprised of cepstral features of a speech signal. In this vector stream, each component represents spectral information from a different frequency band.

Further, various transformations employed in the generation of cepstral features are designed to produce uncorrelated components. To see how vector embedding compares for coupled and decoupled systems, we also experiment on uncorrelated Lorentz time series. This is achieved using the method of Principal Component Analysis (PCA). We estimate the covariance matrix of the three variables in the Lorentz system and compute the transformation matrix that decorrelates the vector series using an eigen decomposition of the covariance matrix. After transformation, we have a set of three uncorrelated variables. Since this is a smooth transformation, we expect all characteristic invariants including Lyapunov exponents to remain unaltered under the transformation. We compare the accuracy of Lyapunov spectra estimates from this uncorrelated vector time series with that of a conventional vector time series of the Lorentz system.

## 5. RESULTS

The estimates of Lyapunov spectra obtained from both scalar and vector embedding of a Lorentz time series as a function of the length of the series are shown in Figure 1. From this figure, it is clear that vector embedding provides reliable estimates (i.e., close to their theoretically expected value) even when the size of the data-set is approximately 2,000 samples. On the other hand, scalar embedding needs at least 8,000 samples for an accurate estimate of the Lyapunov spectra. This indicates that vector embedding provides accurate reconstruction of the system's attractor from a short time series.

Figures 2 through 4 depict the variation in accuracy of Lyapunov spectra estimates of a clean Lorentz series with various parameters. It can be seen that in clean conditions, both scalar and vector embedding provide reliable estimates of the Lyapunov exponents at low values of the algorithm parameters, and the accuracy of the estimates does not vary much with variation in these parameters. With the SVD window size set to 15 and number of neighbors set to 20, we obtain accurate estimates of the Lyapunov spectra.

Figures 5 through 6 illustrate similar results at an SNR of 10 dB. With the SVD window size set to 50 and number of neighbors set to 50, we obtain accurate estimates of the Lyapunov spectra. This follows from the intuition that to estimate dynamical invariants from a noisy trajectory, it becomes necessary to use a larger neighborhood to measure the local dynamics accurately. A larger SVD window size is also required to remove the effects of noise.

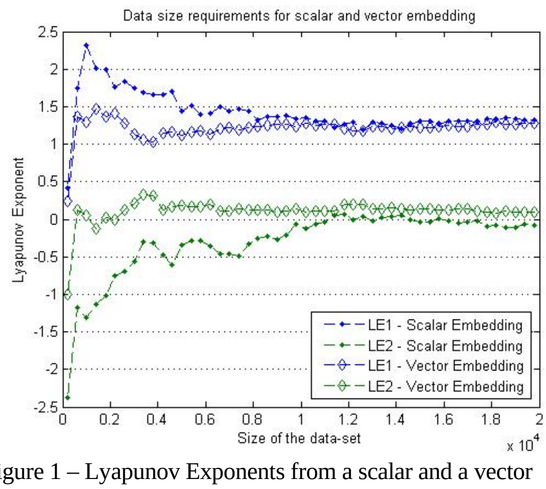


Figure 1 – Lyapunov Exponents from a scalar and a vector (clean) time series as a function of the data-size length

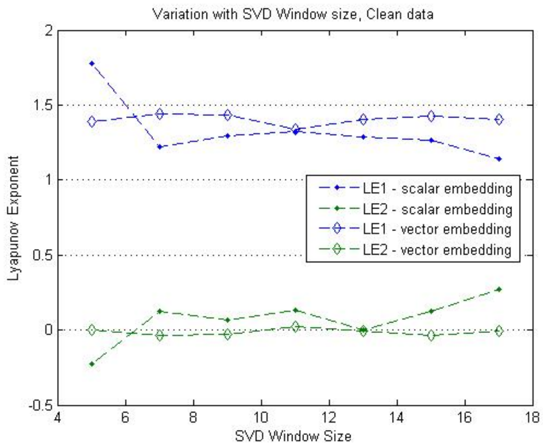


Figure 2 – Lyapunov Exponents from a scalar and a vector (clean) time series as a function of the SVD window length

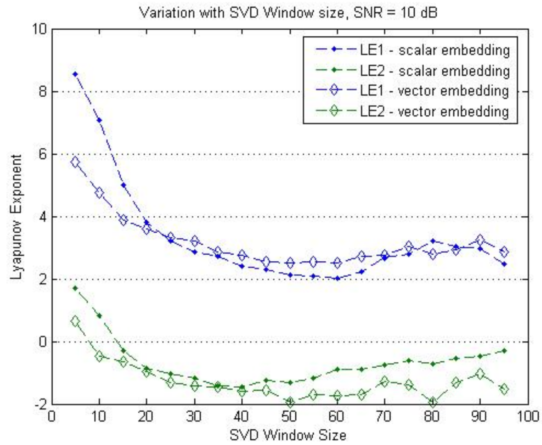


Figure 5 – Lyapunov Exponents from a scalar and a vector time series as a function of the SVD window size, SNR = 10 dB

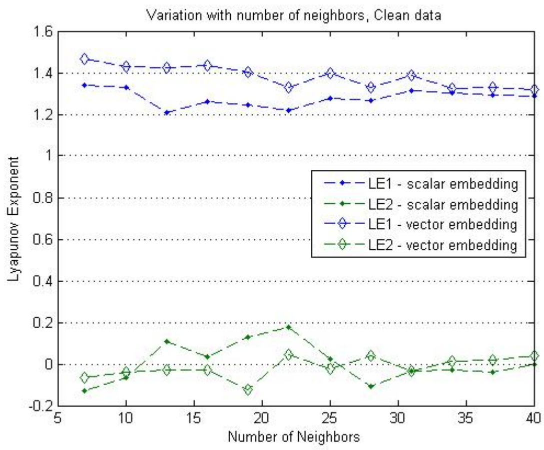


Figure 3 – Lyapunov Exponents from a scalar and a vector (clean) time series as a function of number of neighbors

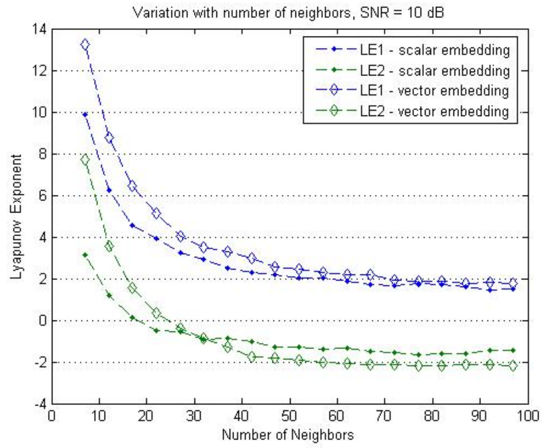


Figure 6 – Lyapunov Exponents from a scalar and a vector time series as a function of the number of neighbors, SNR = 10 dB

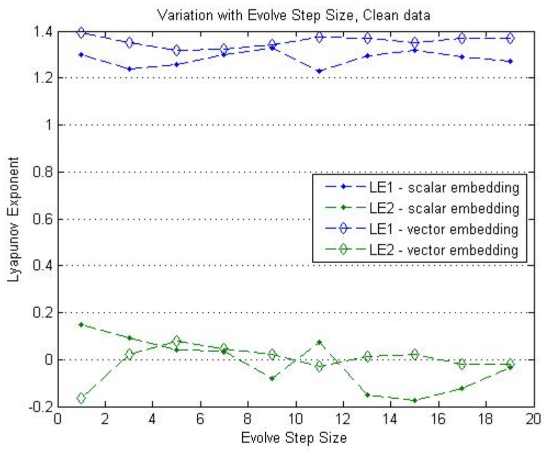


Figure 4 – Lyapunov Exponents from a scalar and a vector time (clean) series as a function of the Evolve Step Size

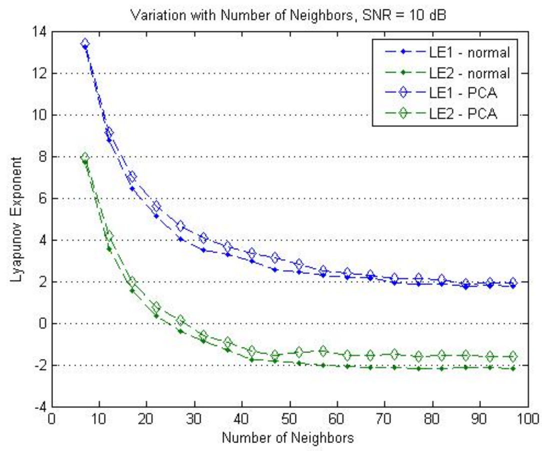


Figure 7 – Lyapunov Exponents from a scalar and a vector time series as a function of the number of neighbors after PCA, SNR = 10 dB

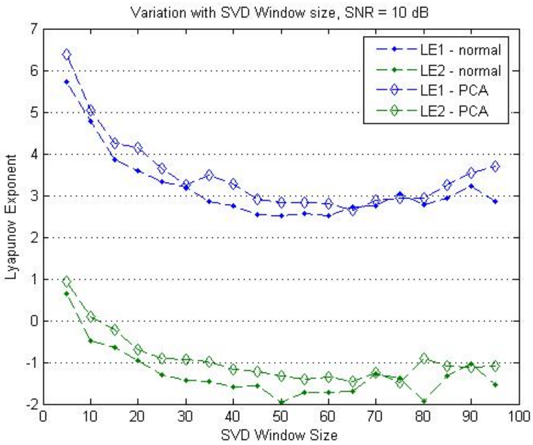


Figure 8 – Lyapunov Exponents from a scalar and a vector time series as a function of the SVD window size after PCA, SNR = 10 dB

Also note that even at a low SNR, the results from vector embedding closely follow those obtained from scalar embedding. Figures 7 and 8 illustrate the Lyapunov spectra estimates of the Lorenz vector time series before and after removing linear correlations among the components of the vector stream. It is clear that de-correlating the stream does not affect the Lyapunov spectra estimates of the system. Although we embedded the vector time series to a final embedding dimension of three, we only report the first two exponents of the spectra (the third exponent is a negative quantity, irrelevant to the analysis of exponential divergence of nearby trajectories).

## 6. CONCLUSIONS AND FUTURE WORK

In this paper, we have shown that extending the concept of phase space reconstruction of a scalar time series to a vector time series comprising of more than one observable of the system provides a reliable reconstruction of the system's attractor. This is illustrated by Lyapunov spectra estimates of a Lorentz system using scalar and vector observables respectively. We demonstrated an important consequence of vector embedding – the data size requirements for accurate estimation of dynamical invariants of an attractor are much less when the attractor is reconstructed from a vector time series, as opposed to a scalar time series. This implies that we can perform meaningful attractor reconstruction using a short time series when vector observables are available.

We also illustrate the efficacy of vector embedding on a data-stream from which all linear correlations have been removed. The fact that we are able to reproduce Lyapunov exponents from an uncorrelated vector stream provides assurance that we can employ the vector embedding technique in many practical situations, where certain pre-processing removes linear correlations from the vector stream (e.g., the Mel Frequency Cepstral stream).

In future work, we plan to use the vector embedding technique on cepstral features for reconstructing attractor trajectories using short data lengths.

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