**COLLEGE OF ENGINEERING**

**Preliminary Exam Report:**

**Quantum Correlations Enabling Quantum Advantage in Machine Learning**

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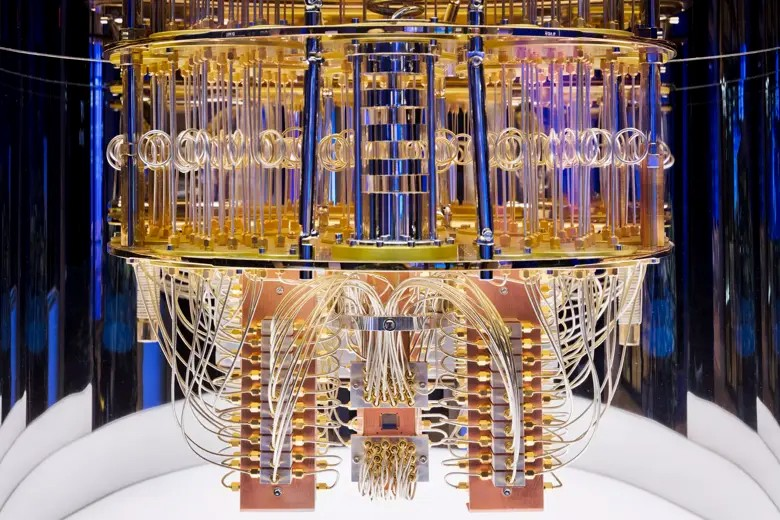
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**Executive Summary**

This report investigates how quantum correlations—specifically entanglement and quantum discord—enable measurable quantum advantages in machine learning systems. It synthesizes findings from three research papers that together establish a theoretical and experimental foundation for understanding how different quantum correlation contribute to performance gains over classical machine learning methods.

The first study, *“Entanglement-Induced Provable and Robust Quantum Learning Advantages”*[1], demonstrates that quantum models using entanglement can achieve exponential speedups in sequence learning tasks. Using the Mermin–Peres magic square game as a benchmark, the authors show that quantum systems can solve specific translation tasks in constant time, while any classical model requires resources that scale linearly with input size. The work further proves that this advantage persists even under moderate quantum noise, revealing entanglement as a computational substitute for classical communication. Despite these advances, challenges remain in extending these results to real-world, continuous data domains and validating them statistically against classical baselines.

The second paper, *“Reformulation of the No-Free-Lunch Theorem for Entangled Datasets”*[2], introduces the concept of entangled datasets, showing that data entanglement can dramatically reduce the number of training samples needed for effective learning. By reformulating the Quantum No-Free-Lunch theorem, the authors prove that stronger quantum entanglement reduces the lower bound on generalization error. In essence, entanglement functions as a resource that increases data efficiency, redefining the relationship between dataset size and learnability. This theoretical insight implies that quantum systems could learn from far fewer examples than classical models, though practical detection and creation of entangled data in real-world scenarios remain open challenges.

The third study, *“The Power of One Clean Qubit in Supervised Machine Learning”*[3], explores the Deterministic Quantum Computing with One Qubit (DQC1) framework, where quantum discord—a weaker form of correlation than entanglement—plays a central role. The authors show that DQC1 can efficiently estimate complex kernel functions for supervised learning tasks using only a single pure qubit, while the rest of the system remains in a mixed state. Experimental results on IBM quantum hardware validate the theoretical model, demonstrating high classification accuracy despite the presence of noise. This highlights quantum discord’s resilience and practicality for current noisy intermediate-scale quantum (NISQ) devices.

Collectively, these studies establish that quantum correlations as a resource for achieving quantum advantage in learning systems. Entanglement provides theoretical and computational breakthroughs by allowing quantum models to outperform classical architectures in complexity and efficiency. Quantum discord, though weaker, enables robust and scalable implementations under realistic noise conditions. Together, these findings bridge the gap between theoretical quantum learning advantages and their experimental realization, offering a roadmap for developing quantum-enhanced machine learning systems capable of outperforming their classical counterparts in speed, data efficiency, and noise resilience.

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# Introduction

Quantum advantage refers to the ability of a quantum system to perform a computational or learning task more efficiently than any known classical method—either in terms of speed, accuracy, resource usage, or sample complexity [4]. In quantum machine learning (QML), quantum advantage manifests when quantum circuits or states allow for faster convergence, smaller training sets, or more expressive models than classical machine learning algorithms can achieve [5].

At the core of quantum advantage lies quantum correlation, a fundamental property of quantum mechanics that describes the relationships between subsystems of a quantum state that cannot be explained by classical probability theory. Unlike classical correlations, which arise from shared randomness, quantum correlations emerge from the inseparability of quantum states. The primary types of quantum correlation are entanglement and quantum discord.

Entanglement is the strongest form of non-local correlation, where the quantum states of two or more particles become intrinsically linked and cannot be described independently, even when separated by large distances. It generates correlations that exceed classical limits and serves as a high-power resource for complex computations [4].

Quantum discord, by contrast, is a more general information-theoretic measure of nonclassical correlation that can exist even in mixed states without entanglement. Although weaker, it is more resilient to noise and environmental decoherence, making it practical for current quantum devices [6].

These correlations serve as the computational fuel of quantum algorithms [7]. Entanglement enables distributed quantum systems to process information collectively, and discord provides noise-resilient correlations that can survive in practical, imperfect hardware. In QML, they translate directly into measurable advantages such as reduced generalization error, requiring fewer training samples, and robustness against decoherence.

Understanding quantum correlations is crucial because they determine whether QML systems can deliver practical advantages over classical methods [5]. Modern machine learning faces fundamental bottlenecks in data efficiency, scalability, and energy consumption—training large models demands vast datasets and immense computational resources. Quantum correlations provide physical mechanisms to overcome these barriers [7].

Entanglement enables distributed subsystems to share information instantaneously without classical communication, potentially reducing both training time and data requirements. Quantum discord, on the other hand, supports stable computation under realistic noise conditions, making it essential for today’s noisy intermediate-scale quantum (NISQ) hardware. Therefore, studying these correlations is not only a theoretical pursuit but also a practical necessity for building quantum architectures capable of outperforming classical algorithms in applications such as pattern recognition, optimization, and generative modeling.

In this report, we examine how different forms of quantum correlations contribute to quantum advantage in machine learning systems. To illustrate this, we have selected three representative papers [1], [2], [3] that demonstrate the theoretical, mathematical, and experimental roles of entanglement and quantum discord in quantum machine learning.

The first paper [1] shows that quantum systems can be much faster and more powerful than regular neural networks for certain tasks. Specifically, it proves that quantum models can perform specific sequence translation tasks in constant time, while classical models need time that grows with the size of the input. This is possible because of entanglement, which allows parts of a system to share information instantly nonlocally —something impossible in classical systems. This ability lets quantum systems avoid the communication limits that slow down traditional neural networks.

The second paper [2] explores how entanglement affects learning itself. It reformulates the Quantum No-Free-Lunch Theorem (NFL) to show that stronger entanglement (measured by the Schmidt rank) helps reduce the minimum possible learning error. In simple terms, more entanglement means we need far less training data to learn well. This solves one of the biggest problems in quantum machine learning: the need for huge amounts of data. In this way, entanglement acts like a valuable "currency" for improving learning efficiency.

Lastly, the third paper [3] studies a special model called Deterministic Quantum Computing with One Qubit (DQC1). It shows that quantum discord, a weaker quantum correlation, can be useful. Discord helps quantum systems handle noise better and make efficient measurements. This is practical for today’s NISQ devices, which are powerful but still limited and noisy [8].

In summary, these papers establish a hierarchy of quantum resources to get quantum advantages in machine learning systems. Entanglement gives fundamental complexity breakthroughs, redefining the theoretical limits of learning and computation. Quantum discord, in contrast, provides robust, noise-tolerant mechanisms essential for implementing quantum learning on current hardware. Together, these findings bridge the gap between the theoretical and practical frontiers of quantum machine learning, highlighting how different quantum correlations serve distinct but complementary roles in advancing scalable, high-performance quantum intelligence.

# Quantum Computing

Before exploring advanced quantum learning and resource theories, it is essential to understand the foundational concepts of quantum states, qubits, and the formal definition of quantum correlations such as entanglement and quantum discord. These ideas form the conceptual framework for the discussions in the subsequent paragraphs. Table 1 describes some common symbols used in quantum computing literature.

A qubit (quantum bit) is the fundamental unit of quantum information, analogous to the classical bit [4]. While a classical bit can only exist in one of two distinct states, 0 or 1, a qubit can exist in a superposition of both. A simple physical example is an electron with two energy levels:

|  |  |
| --- | --- |
| **Symbol** | **Meaning** |
| |ψ⟩, ⟨ψ| | Quantum state in bra–ket notation |
| |ψAB⟩ | Multi-qubit system where |ψAB⟩ = |ψA⟩ ⊗ |ψB⟩ |
| ρ | Density matrix representing mixed quantum state |
| U | Unitary operator representing quantum evolution |
| U† | Conjugate transpose of the unitary operator, U |
| ⊗ | Tensor product of quantum systems |
| ⟨A⟩ | Expectation value of operator A |
| U|x⟩ | Applying the unitary operator U to the quantum state |x⟩, which transforms the state into a new quantum state |
| ℋ | Hilbert space of the system |
| Tr(ρ) | Trace of density matrix; equals 1 for normalized states |
|  | Probability amplitude or state coefficient |

Table 1: Common symbols and its meaning.

* The ground state (lower energy) represents |0⟩.
* The excited state (higher energy) represents |1⟩.

From the Heisenberg uncertainty principle [9], we know that it is impossible to simultaneously determine an electron’s exact position and energy with absolute precision. This fundamental uncertainty implies that the electron cannot be said to occupy a single, definite energy state at all times. So, if the electron is in the ground state with probability √α and in the excited state with probability √, its overall state is described as a probabilistic quantum state. Conceptually, this electron can represent a bit of information through its energy configuration, and has two possible qubits —|0⟩ or and |1⟩ or . So, we can write the state of one electron as follows:

The above equation represents a quantum state that encapsulates all the information about a physical system, in this case, an electron. Mathematically, a quantum state is described by a state vector (denoted as |ψ⟩) in a Hilbert space (ℋ). The probability of observing the system in a particular state (|0⟩ or |1⟩) is determined by the squared amplitude of that state’s coefficient in the superposition. |0⟩ and |1⟩ are also called basis qubit states. When we observe or measure a quantum state with respect to any basis vector (|0⟩ or |1⟩, Pauli , Pauli , etc.), the quantum state collapses. This means that quantum properties such as superposition and entanglement vanish, and we are left with the classical probability of the system being in a particular measurement basis state.

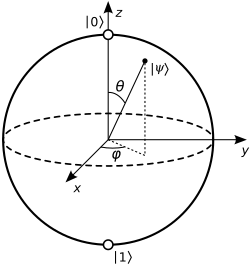


Figure . A bloch sphere

The Bloch sphere [10] shown in Figure 1 allows for the understanding of additional representations of the qubit state-vectors as they provide the conversion of the vector-state from the two-dimensional complex-space to a three-dimensional real vector space using two angles and .

In quantum mechanics, pure states and mixed states help describe how much we know about a quantum system. They also tell us what kinds of quantum correlations (like entanglement and discord) the system might have. A pure state is one that can be described by a single state vector (|ψ⟩) or equivalently by a density matrix ρ = |ψ⟩⟨ψ|. It represents a system with complete information. For example, an electron definitely in the ground state, |ψ⟩ = |0⟩. Mathematically, a pure state satisfies ρ² = ρ and Tr(ρ²) = 1, meaning that the system is entirely coherent and there is no statistical uncertainty about its state.

In contrast, a mixed state represents incomplete knowledge of the system. It is a statistical ensemble of several pure states, expressed as ρ = Σᵢ pᵢ |ψᵢ⟩⟨ψᵢ|, where each state |ψᵢ⟩ occurs with probability pᵢ and the probabilities sum to one. For mixed states, ρ² ≠ ρ and Tr(ρ²) < 1, indicating that the state is not fully coherent and includes uncertainty.

The connection between these states and quantum entanglement becomes clear when we consider composite systems, meaning dealing with two or more qubits. Entanglement arises when the combined state of two subsystems cannot be written as a simple product of their individual states. In a pure bipartite system, entanglement can be identified by checking whether the total pure state |ψAB⟩ can be expressed as a product |ψA⟩ ⊗ |ψB⟩. If this factorization is impossible, the system is entangled. A common example is an EPR pair, short for the Einstein-Podolsky-Rosen pair, named after the physicists Albert Einstein, Boris Podolsky, and Nathan Rosen. An EPR pair is typically represented by the Bell state [11] defined as which cannot be written as a simple tensor product of individual qubit states. This inseparability means that the two qubits share a single joint quantum state-measurements on one immediately affect the state of the other, no matter how far apart they are.

A useful way to think about quantum entanglement is by comparing it to shared variables in classical computation—but with a crucial difference. In a classical computer program, a static variable represents a single shared memory location. Whenever one function updates this variable, every other part of the program accessing it immediately sees the new value. This reflects a form of shared reference, where all functions depend on the same physical memory address.

Quantum entanglement might appear similar at first glance: when two particles are entangled, measuring one instantly determines the state of the other. However, unlike a static variable, the two particles do not share a physical connection or a common memory location. Instead, their properties are correlated in a way that cannot be explained by any local mechanism. This means that even if the particles are separated by light-years, a measurement on one will immediately define the outcome of the other.

This phenomenon, called non-local correlation, distinguishes quantum systems from classical ones. In classical systems, correlations require communication or shared memory; in quantum systems, entangled particles behave as a single, inseparable system no matter how far apart they are. This non-locality is one of the most striking and counterintuitive features of quantum mechanics and a key resource for quantum technologies.

While entanglement captures only a subset of all possible quantum correlations, quantum discord accounts for all nonclassical correlations in a system. In pure states, quantum discord and entanglement are equivalent, since all correlations are purely quantum in nature. However, in mixed states, even separable (non-entangled) states can possess nonzero quantum discord, meaning they still exhibit some level of quantum behavior. This occurs because a measurement on one subsystem can disturb the state of the other, revealing an underlying quantum structure. For instance, the state is a single-qubit superposition state defined as The factor (and equivalently the factor in two-qubit expressions like ) ensures the state is normalized, meaning the total probability of finding the qubit in either or is 1 . It also indicates that both basis states contribute equally-this equality of amplitudes is what makes a genuine superposition of and , rather than a probabilistic mixture. So, the mixed state defined as where the state means both qubits are in the state which can be expressed as a tensor product:

Each component ( and ) is a product state, so their mixture is separable and contains no entanglement. However, because and are not orthogonal, measuring one qubit can still disturb the overall state. This measurement-induced disturbance gives rise to nonzero quantum discord, revealing that even separable states can exhibit genuinely quantum correlations rooted in superposition [7].

The fundamental principles of superposition, entanglement, and quantum discord not only define the structure of quantum systems but also determine their computational potential. With this conceptual foundation, we now turn to how these correlations—particularly entanglement—translate into measurable advantages in quantum learning tasks.

# Quantum Entanglement-based Advantage

This section analyzes how entanglement enables provable computational advantages in quantum learning systems. Many earlier studies suggested possible advantages, but these often depended on fault-tolerant quantum computers that are not yet available and remained mostly theoretical with limited experimental validation. The authors of the first paper [1] take a clear step forward by creating a simple learning task where a small quantum model works much better than any classical model using quantum entanglement.

The main contribution is that they design a natural sequence-to-sequence learning task for which a constant-depth, constant-parameter quantum model succeeds decisively, while any standard classical sequence model must scale its internal resources linearly with the input length to avoid exponentially small accuracy. The authors further require two features that prior constructions often lacked: (i) the advantage should survive constant physical noise (not merely levels), and (ii) the optimal quantum model should be trainable with constant data and time—a complete, end-to-end story from problem design to learnability and experiment. In summary, the quantum model can be trained quickly and still performs well even with some noise.

The narrative begins with a non-local game is an interactive protocol between some players and a verifier. We focus on the case where there are only two players and . The verifier gives each player a query , and require it to output an answer . It was assumed that the queries are independently sampled from the uniform distribution over . The rule (or winning condition) of the game is specified by a predicate function , where indicates winning.

The players can coordinate with each other to set up a strategy for producing the answers, but they cannot communicate anymore after receiving the queries (i.e., a player does not know what the queries of the other players are). A classical strategy is a strategy in which players can access a shared classical source of randomness and prior information to make the plan. In a quantum strategy, players can access a shared (entangled) state and perform quantum operations on it to output the answers. The maximal winning probability of classical (quantum) strategies is called the classical (quantum) value of the game. We use and to denote the classical and quantum values of the game , respectively.

**Table 2.** A classical strategy (left) and a quantum winning strategy (right) of the magic square game [1].



A game is a quantum pseudo-telepathy game if it has quantum value one but classical value strictly smaller than one. In other words, a quantum pseudo-telepathy game can be won by a quantum strategy with certainty, but requires a certain amount of communication to be won perfectly by any classical strategy. Such games have found applications in proving that entanglement can be used to reduce the classical communication burden of certain distributed computational tasks. This reduction of classical communication is not via transmitting information with entanglement, which would have violated special relativity. Instead, it is achieved by allowing quantum protocols which are more general than classical strategies.

In this work, the authors extensively use a quantum pseudo-telepathy game called the Mermin-Peres magic square game. It is a two-player game where the queries and the answers . The game is won if all of the following three conditions are satisfied:

where is the addition over . These conditions force the answers of the players upon receiving each possible queries (i.e., their strategy) to form a square, as shown in the left panel of Table S1. In the square, the -th column represents the answer given by upon receiving the query , and the -th row represents the answer given by upon receiving the query . Each column must sum to 1 , and each row must sum to 0 . But this is classically impossible, as it means that the sum of all numbers in the square must be odd and even simultaneously. Therefore, for any classical strategy, there must be at least one case in all the cases where the game is not won. This means that the classical value of this game is .

On the other hand, there is a quantum strategy that wins the game with certainty, as shown in the right panel of Table 2. The two players share an entangled state

where holds register and , and holds the rest. Upon receiving queries , they produce answers by measuring the observables in the -th column and -th row and output the measured bitstrings. Note that in any column or row, the observables commute with each other. Therefore, they can indeed be measured simultaneously and produce length- 3 bitstrings .

**Table 3.** Illustration of the magic square translation sub-task rule . For each possible input sequence . we list all the valid translation outputs such that . Here, \* can be any single bit.

|  |  |
| --- | --- |
| input | valid translation outputs (i.e., ) |
| 00\*\* | \*\*\*\* |
| \*\*00 | \*\*\*\* |
| 0101 |  |
| 1001 |  |
| 1101 |  |
| 0110 | *00*, *11* |
| 1010 |  |
| 1110 | \*001, \*010, \*100, \*111 |
| 0111 | 010\*, 100\*, 001\*, 111\* |
| 1011 |  |
| 1111 | 0101, 1010, 0000, 1111 |

In the quantum magic square strategy, the pattern of how observables (, ) multiply within each row and column reveals the essential algebraic asymmetry that enables perfect quantum correlations. Each observable in the table is a tensor product of Pauli matrices acting on two qubits-for examples, and . When we examine a row, such as the first one containing , and , their product is , which simplifies to . Hence, every row multiplies to the identity, corresponding to an eigenvalue of +1 and representing even parity in measurement outcomes. In contrast, when we examine a column, for example, the first column consisting of , and , the product becomes . Thus, each column multiplies to , giving an eigenvalue of -1 and indicating odd parity. Together, these relations rows yielding and columns yielding -ensure that two qubit’s measurement outcomes satisfy the parity constraints of the magic square game. The carefully arranged sign structure of these commuting observables is what allows quantum strategy to achieve perfect correlations that are impossible in any classical deterministic model. This also means that the quantum strategy in Table 2 can win the game with certainty using two entangled pairs, and the quantum value of the game is one.

In the paper, the authors adopted a variant of the magic square game where the input and output formats are regularized. Specifically, let , , and and define,

The game is won if and only if at least one of the following two conditions are satisfied: (1) either or is 00; (2) . We immediately see that the non-trivial part of the game (i.e., when neither of is 00) is equivalent to the magic square game. Therefore, it has quantum value and classical value .

To convert a non-local game into a supervised translation task, the authors define a sub-task , that accepts precisely the () pairs corresponding to winning transcripts of An 𝑛-fold parallel repetition then forms the full “magic-square translation” task with the effect that classical non-communicating success decays as , while the entangled strategy wins every round.

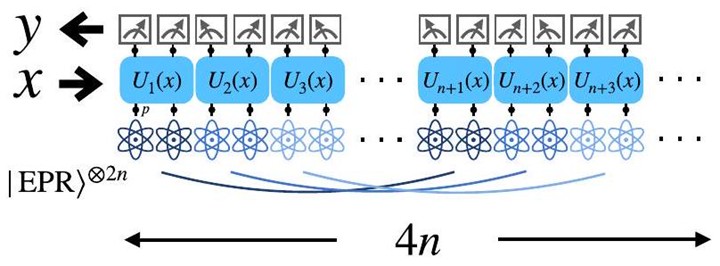


Figure 2. A quantum model [1]

The quantum model is a single-layer, weight-shared Clifford circuit acting on a qubits initialized as EPR pairs. Two local, input-conditioned two-qubit unitaries rotate the magic-square observables so that computational-basis measurements directly produce the required output bits Because the same tiny gate set is reused across all sub-tasks (weight sharing), the parameter count and depth remain , yet the models wins perfectly on . The authors provide three theorems as follows:

**Theorem 1 (Noiseless inference advantage):** There exists a magic-square translation task such that an -parameter, -depth quantum model using Bell pairs achieves score 1, while any classical model achieving nonexponentially small score must have parameters.

Split the input in half and bound the input-dependent communication capacity a classical model needs to couple the two halves. A transcript-guessing reduction converts any communicating model into a non-communicating strategy, so its score () follows:

For common sequence models (autoregressive, encoder-decoder), information subadditivity yields for model size/latent dimension ; thus achieving requires . The quantum model embeds the entangled winning strategy and inherits perfect score.

**Theorem 2 (Inference advantage with constant noise):** Under single-qubit depolarizing noise of strength after each gate, for all there exists a fractional-win variant for which an -parameter still achieves , while any classical model needs parameters to reach score.

It is assumed that the optimal sub-circuit for has 10 potential noise locations, giving a per-round noisy quantum value . Choosing a threshold

and accepting when at least a fraction of sub-games win, standard concentration inequalities [12] ensure the quantum score concentrates above (hence ), while classical success still decays like . The separation persists whenever , i.e. .

**Theorem 3 (Training advantage):** There exists a maximum-likelihood training algorithm which, with probability at least , outputs the optimal using training data of constant total size (equivalently, samples) and constant time .

Theorem 3 concerns training. Because the quantum hypothesis class has only parameters, a simple maximum-likelihood search recovers the optimal model using constant-size training data and constant training time; informally, as grows each labeled pair ( ) carries bits of evidence, so the required number of samples shrinks like . In contrast, the classical side provably needs parameters for comparable accuracy, which inflates its hypothesis space and typical sample needs.

The paper links five steps. First, it seeds a constant per-round gap: the regularized magic-square game provides with a tidy input-output format suited to building a dataset. Next, the authors amplify by parallel repetition: bundling independent copies yield a translation task whose classical non-communicating success probability decays as while the quantum strategy remains this is the game-value lever used in Theorem 1. The paper then turns accuracy into model size: a communication-capacity bound converts the value gap into a linear lower bound on the number of classical parameters , giving a constant-vs-linear inference separation (Theorem 1). To make the result robust to real noise, the authors quantify the noisy quantum value per sub-game as and relax the acceptance rule to a calibrated fractional win ; concentration keeps the quantum advantage for all .(Theorem 2). Finally, the paper closes the loop with learning: because the quantum hypothesis class is finite and constant, maximum-likelihood estimation needs only constant samples and time (Theorem 3), whereas classical models with parameters require samples-hence -as suggested by standard generalization bounds.

Numerical experiments show (Figure 3) the classical score at fixed hidden size decays roughly like consistent with the communication-capacity analysis, while the quantum model maintains high scores under realistic noise. Runs on lonQ's 25-qubit trapped-ion machine (Aria) achieve an average score above 0.88 for problem sizes up to on the noisy task with a 0.95 fraction threshold, despite device noise not yet meeting the theoretical . The authors argue performance should improve as hardware noise drops, aligning with Theorem 2.

The paper isolates entanglement as a resource that substitutes for classical communication. In the streaming translation task, it removes the need to pass messages along the sequence during inference; classical models can only mimic that by carrying a bigger and bigger hidden state—hence the growth. The noise bound is not vanishingly small. Each sub-task uses a fixed handful of gates, so the chance of a noisy failure per sub-task is bounded by a constant. Requiring a fixed fraction of wins across 𝑛 sub-tasks gives the quantum side room to tolerate those constant-rate glitches, while classical models still face the same communication wall. Also, training is easy on the quantum side as fewer parameters mean fewer hypotheses to distinguish; longer sequences pack more information per labelled example. That is the rare learning problem where bigger inputs make training easier, not harder.

*From a critical standpoint*, the paper presents a compelling theoretical framework proving that quantum entanglement can accelerate both training and inference, even in the presence of quantum noise. However, its formulation relies on a simplified input model where data are represented as discrete binary values (0 or 1). In contrast, real-world data are typically continuous and high-dimensional, raising the question of how the proposed theorems would generalize beyond this idealized setting. Furthermore, the work assumes the use of pre-computed EPR pairs, but it does not address how classical continuous data could be efficiently encoded into such quantum states. This highlights a significant gap between theoretical feasibility and practical implementation. Contemporary large language models, for instance, employ embedding dimensions ranging from 768 to 4096 [13], [14], and directly encoding such spaces into qubits—whether through angle encoding (requiring one qubit per feature) or amplitude encoding (logarithmic in dimension) [15]—poses serious scalability and noise challenges, particularly on NISQ devices. Consequently, future research should investigate how entanglement-based acceleration behaves when extended to realistic, continuous input domains and how robust such methods remain as qubit counts and noise levels increase.

|  |  |
| --- | --- |
| **Figure 3.** Results on numerical simulations and trapped-ion experiments on IonQ Aria [1]**.** |  |

Additionally, the paper does not include any form of statistical significance analysis with its classical counterpart algorithms. It is also important to acknowledge that demonstrating quantum advantage cannot rely solely on traditional significance testing, such as p-value analysis. While p-values are useful for assessing the likelihood that observed differences arise by chance, they are insufficient to capture the full statistical and practical meaning of a quantum–classical performance gap. In the context of quantum computing, establishing statistical significance requires additional, domain-specific metrics—such as fidelity, circuit depth efficiency, resource overhead reduction, or speedup scaling relative to noise levels—to determine whether a quantum algorithm is meaningfully better than its classical counterpart. Therefore, future research should not only extend the theoretical framework to handle continuous and high-dimensional data but also develop comprehensive statistical and empirical evaluation methodologies capable of rigorously quantifying quantum advantage under real-world computational and noise constraints.

In summary, the first study demonstrates that quantum entanglement acts as a direct substitute for classical communication, enabling efficient and robust learning even under noise. Yet, entanglement’s role extends beyond computation—it can also reshape how data itself contributes to learning. The next section explores this broader perspective by examining how entangled datasets alter the theoretical limits of generalization in machine learning.

# Entangled Data and Quantum NFL

Building on this foundation, the second study [2] extends the notion of entanglement from computation to data itself. They ask a central question of learning theory and its quantum extension: if classical “no-free-lunch” limits say that, on average, we cannot beat data scarcity by choosing a clever optimizer, does quantum mechanics change what “enough data” means once training examples may be entangled with a reference system. The authors formulate a supervised learning setting in which the task is to learn an unknown unitary map using quantum training pairs and show that allowing the training states to be entangled yields a quantitatively different and strictly more permissive no-free-lunch bound. In effect, entanglement becomes a fungible currency so the budget for learnability is not only how many examples we have but also how entangled those examples are.

The setup replaces the usual supervised-learning tuple of inputs and labels with pairs of quantum states. The unknown target is a d-dimensional unitary . A training example is a state on the input system together with a reference , and its "label" is the output state obtained by applying to while doing nothing to . The learner searches over a hypothesis unitary that exactly interpolates the training set, meaning reproduces every training pair perfectly. The key new knob is that can be entangled across and , and the amount of entanglement is summarized by its Schmidt rank , which ranges from 1 (no entanglement) to d (maximal entanglement). The authors define the generalization error, or "risk," as the average squared trace distance between the true output and the hypothesis output over uniformly random pure inputs on X. This risk is directly tied to the average output-state fidelity via a standard identity: if , then the risk equals . This formula converts the performance question into a group-integral question about when is averaged over random targets .

The first mathematical insight is that perfect interpolation on the training set constrains the mismatch unitary to a block-diagonal form. If the training inputs each have Schmidt rank , then those constraints pin down an -dimensional subspace on which acts as a global phase , while the unconstrained -dimensional orthogonal block is an arbitrary unitary . Concretely, after choosing a basis adapted to the training subspace, looks like dlag with r copies of per training pair. The trace then splits as . When we average over random targets , the randomness transfers to the free block , and the Weingarten calculus shows that while cross terms vanish by unitary-invariance. Hence . Plugging this back into the risk identity yields the central bound

The average risk is therefore controlled by the product . When there is no entanglement, , we recover earlier quantum NFL bounds that require to drive the average risk to zero, which is generically exponential in the number of qubits. When the examples are maximally entangled, , a single training pair is enough in principle to make the lower bound vanish, which matches the intuition that one Choi state uniquely identifies a quantum channel. The theorem is tight in that the inequality is saturated whenever the training inputs are linearly independent, so the bound is not merely a pessimistic artifact of averaging.

The authors tested their theory both on actual quantum hardware and in computer simulations. On Rigetti’s Aspen-4 quantum computer, they trained small quantum operations (unitaries) using two types of training data: one with no entanglement between quantum states () and one with entangled states (). They found that, with the same number of training examples, the entangled data led to much lower average error (risk) and matched the theoretical predictions closely. The small differences they observed were mainly due to noise and limited data.

They also ran larger simulations (Figure 4-5) on six qubits, varying the number of training examples ( from ) and the amount of entanglement ( as powers of two). The results closely followed the theoretical formula. The plots clearly show the main idea: when the product of and (that is, ) is the same, the data curves align; and when is larger, the error drops faster toward zero.

|  |  |
| --- | --- |
| Figure . Classical and quantum NFL bounds showing reduced risk with increasing entanglement. | Figure . Quantum hardware results showing entangled data gives lower risk than unentangled data. |

The notion of an "entangled dataset" becomes precise in this framework. A dataset is a set of input-output pairs where each input lives on and has Schmidt rank, across the split . The rank measures how many orthogonal "branches" of correlation between and the example contains. Rank one means a product state with no correlations, while rank means a maximally entangled state that distributes amplitude evenly across a basis on X and a conjugate basis on . Because training demands that maps every to the correct output , each entangled example knocks out not just a single direction on but an -dimensional slice, and such examples carve out a total constrained subspace of dimension up to . An "entangled dataset" therefore literally amplifies the information per example by its Schmidt rank, which is why the product is the quantity in the bound.

*From a critical perspective*, the finding that a higher degree of entanglement within a dataset implies that less data is required for learning is an extremely significant and potentially disruptive development for current machine learning research. This result challenges the data-hungry nature of classical machine learning systems by suggesting that quantum entanglement could serve as a new form of data efficiency. However, a major research gap remains in determining how to identify whether a real-world dataset is entangled. While entangled datasets can be simulated in laboratory settings or through numerical models, discovering entangled samples in real-world data—such as EEG signals or multimodal datasets including video and sensor data—would represent a major step forward.

One possible approach involves measuring quantum correlations using metrics such as the CHSH Bell value [16]. If the expectation value of a quantum system exceeds two, this indicates a type of correlation that cannot be captured using classical measures. Detecting such quantum-like properties in real-world data could provide evidence that certain datasets exhibit entanglement-like structures. If such properties exist, the implications are profound: according to the quantum NFL theorem, systems that leverage entangled data would require significantly fewer samples to achieve comparable generalization performance. This would render current classical machine learning algorithms, which depend on massive amounts of data, obsolete.

Nonetheless, challenges remain. The current formulations of the quantum NFL theorem assume idealized conditions of perfect training—meaning all hypotheses are correctly predicted. In practice, however, such perfect training conditions are exceedingly rare in machine learning systems. Real-world data are noisy, incomplete, and often unbalanced, making perfect optimization nearly impossible. Therefore, future theoretical work should aim to develop a generalized form of the quantum NFL theorem that accounts for imperfect training conditions and noisy data environments.

While entanglement provides the strongest form of quantum correlation and enables profound theoretical shifts in data advantage, it is also fragile under noise and decoherence—conditions that dominate current quantum hardware. This limitation motivates the exploration of weaker yet more robust correlations. The next section investigates quantum discord, a subtler form of correlation that can sustain quantum advantage even in noisy, mixed-state environments.

# Discord-based Advantage

Finally, the third study [3] explores how quantum discord—though weaker than entanglement—provides practical advantages for noisy quantum devices. The authors explore whether the non-universal "power of one clean qubit" model (DQC1) can do useful work for supervised learning by estimating kernels efficiently and robustly, and it answers in the affirmative by tying the learnability of data directly to coherence consumption and quantum discord while demonstrating the full pipeline on IBM hardware.

The core machine-learning problem is to evaluate a kernel fast and accurately enough that a classical SVM trained on that kernel generalizes well; when the kernel has the right structure, computing it classically can be costly, so a quantum algorithm that estimates the kernel more efficiently, with fewer or less fragile quantum resources, is valuable. The authors show that DQC1-which uses a single, partially pure control qubit coupled to a register in a maximally mixed state can estimate a broad family of kernels by encoding the pair () into a unitary and reading out from the Bloch vector of the control qubit, so that the offdiagonal expectation of the control qubit equals the complex kernel value. Because DQC1 achieves this with only one measured qubit and without entangling the clean qubit with the mixed register, the method is naturally resilient to the kinds of noise that most strongly affect multi-qubit measurements on today's devices.

The DQC1 model reveals the kernel function through a mathematically straightforward process. The control qubit starts in with a tunable purity, and the target qubits start maximally mixed, then a Hadamard on the control and a controlled- produce a joint state whose reduced control density matrix is . Measuring and *Pauli-* on the control recovers the real and imaginary parts of within additive error using shots independent of , which is the familiar DQC1 trace-estimation scaling and is the crux of the computational appeal. This scaling separates the intrinsic statistical cost of estimation from system size, and it holds even though efficiently approximating DQC1 output probabilities in general is believed to be classically intractable. The authors also explained that in DQC1 the control and register remain separable throughout the computation, which explains the protocol's robustness to entanglement destroying noise while still harvesting nonclassical correlations.

The definition of quantum coherence is it describes a system's ability to maintain a stable, synchronized phase relationship between its quantum states, allowing it to exist in multiple states simultaneously through superposition. The coherence of the control qubit, quantified by the relative-entropy coherence defined as , is reduced by running the controlled- . The amount consumed can be determined by for , where is the binary entropy. This formula immediately yields two operational interpretations that are very useful.

First, if no coherence is consumed when comparing to , then , which means the feature states are indistinguishable to the chosen map and the classifier cannot learn to separate them; conversely, larger coherence consumption implies greater discriminative power. Second, on real hardware the diagonal kernel elements should satisfy , and any deviation diagnoses stochastic noise or coherent errors as literal lost coherence, which turns a messy hardware effect into an interpretable knob in the learning pipeline. The authors validate this relation experimentally.

The role of quantum discord is made concrete through the geometric discord of the control-register state after the circuit, which has a closed form in DQC1: . The authors show that the discord generated during the run is upper bounded by the coherence consumed from the control qubit, , so discord is literally created by spending coherence. This inequality "makes sense" physically because the controlled- converts local superposition on the clean qubit into nonclassical correlations with the mixed register without ever creating entanglement, and mathematically because geometric discord in DQC1 depends on the same trace structure as the off-diagonal control elements that define the coherence loss. The same measurement strategy that estimates can be repeated to estimate , which lets one quantify discord from experimental data and verify the bound empirically. The key advantage of learning is that discord, unlike entanglement, is more tolerant to realistic noise channels, so a DQC1 kernel can remain useful in regimes where entanglement-based kernels degrade.

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| Figure 6. The plot shows accuracy versus control qubit purity (α): when α = 0 the state is fully mixed, when α = 1 it is pure. | Figure 7. Simulation results of learning process for “make-moon” and “make-circle” dataset |

The paper instantiates these ideas with a complete supervised-learning workflow. The data are embedded via the Havlíček-style diagonal feature map [17] layered as , and the kernel is defined as . For and , the authors implement the DQC1 circuit on IBM's seven-qubit "ibm\_perth" device using only the control qubit to build the kernel matrix for training and to compute test-time kernel evaluations; with identical encoding on a simulator, they obtain perfect classification on the "ad\_hoc" dataset [17], while hardware achieves accuracy with errors attributable to noisy gates and readout. The authors further sweep the control purity and observe the expected transition (Figure 7) from chance-level accuracy at up to high accuracy beyond a dataset-dependent threshold, which matches the theoretical reading of as a learnability meter and shows how "how much coherence we are willing to spend" controls performance. They visualize the simulator and hardware kernels (Figure 8) and the derived coherence-consumption and discord heatmaps (Figure 9). These plots confirm the deviation along the diagonal and the inequality between generated discord and consumed coherence.

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| Figure 8. Heatmap of coherence consumption. | Figure 9. Results for the DQC1 quantum kernel. |

The theoretical statements in the paper align with standard complexity and information-theory intuitions while remaining consistent with known caveats for quantum kernels. The DQC1 trace-estimation function uses standard concentration to yield the shot complexity, where is the target precision. This explains why measuring only the control qubit is enough and why the sampling cost does not increase with system size—the uncertainty in the estimate depends on the value of the trace being measured, not on the number of qubits in the system.

Although each kernel entry is size-independent to estimate to fixed precision, the authors note that for many structured encodings the kernel values themselves can concentrate exponentially with the number of qubits, which implies that the number of shots needed to resolve useful variation may grow quickly; in such regimes the untrainability phenomena identified for fidelity-based kernels can appear here as well. The hardware results quantify noise through the deviation of diagonals from unity and through the accuracy-versus-purity curves, which give actionable diagnostics for choosing , compiler synthesis fidelity, and circuit depth. The comparison to projected kernels shows that measuring a single clean qubit can be competitive in accuracy while saving a large constant measurement overhead and avoiding fragile multi-qubit tomography. The broader message is that in noisy intermediate-scale devices, discord can be a practical workhorse while entanglement is scarce, so DQC1 offers a principled path to useful quantum kernels with a small, interpretable resource budget.

*From a critical perspective*, this paper marks an important step toward achieving quantum advantage using only one clean qubit. It demonstrates that complex kernel functions for supervised learning can be efficiently estimated by exploiting quantum discord and quantum coherence. Unlike earlier studies, the authors designed a binary classification task implemented on quantum hardware with a toy dataset containing continuous values. The experiment achieved high accuracy, highlighting quantum discord as a more noise-resilient alternative to entanglement in real quantum devices. Another important distinction from the other two papers is that this study discusses how to encode classical data into quantum states using the ZZFeatureMap [17]. Also, the authors study the coherence consumption rate, which can be considered an evaluation metric specific to the quantum model and which proves the robustness of quantum discord.

Although the DQC1 model demonstrates that quantum discord can provide measurable learning advantages using minimal resources, its performance did not compare with other quantum machine learning frameworks. In contrast to Variational Quantum Classifiers (VQC) [18], which rely on parameterized quantum circuits trained through gradient-based optimization, DQC1 does not require iterative training and thus avoids issues like barren plateaus and high gradient variance. This makes DQC1 more stable and efficient on NISQ hardware. However, VQCs offer greater expressive power through tunable parameters and deeper circuits, which can model more complex decision boundaries when sufficient qubits and noise control are available. The study also lacks a comparative analysis with classical support vector machine kernel functions and does not report statistical significance for the results.

In addition, compared to Quantum Kernel Estimation (QKS) frameworks [19]—which evaluate kernel functions using entangled feature maps—DQC1 achieves kernel estimation with substantially fewer resources by leveraging a single clean qubit and exploiting quantum discord rather than entanglement. This leads to improved noise tolerance but can limit scalability for large datasets. Therefore, while DQC1 provides a promising low-resource path to near-term quantum advantage, it complements rather than replaces variational and kernel-based approaches, each occupying distinct positions on the trade-off between expressivity, robustness, and scalability. Future work could integrate these paradigms—for example, hybrid architectures that combine DQC1’s discord-driven kernel estimation with VQC-based adaptive parameterization—to achieve both noise resilience and expressive capacity. Future work could integrate these paradigms—for example, hybrid architectures that combine DQC1’s discord-driven kernel estimation with VQC-based adaptive parameterization—to achieve both noise resilience and expressive capacity.

# Future Research Directions

The findings discussed in this report collectively demonstrate that quantum correlations—entanglement and quantum discord—serve as the fundamental enablers of quantum advantage in machine learning. Each form of correlation offers distinct benefits: entanglement enables computational speedups and data efficiency by allowing distributed quantum systems to act collectively, while quantum discord ensures stability and robustness in the presence of noise. Together, they form a complementary resource hierarchy for advancing quantum machine learning systems across both theoretical and practical domains.

Entanglement provides the strongest form of quantum correlation and remains central to the demonstration of quantum advantage. The first study examined shows that entangled quantum systems can achieve constant-time solutions to problems that require linear resources in classical models. This advantage arises because entanglement effectively replaces the need for explicit communication between system components, allowing qubits to share information nonlocally. However, the current models assume discrete binary datasets and perfect quantum hardware, leaving open questions about how entanglement-based learning would scale with continuous, high-dimensional, or noisy data. Extending these theoretical frameworks to handle realistic data distributions remains a necessary step toward demonstrating practical quantum learning advantages.

The reformulated Quantum NFL theorem redefines the limits of generalization by introducing entangled datasets as an informational resource. The key insight is that the degree of entanglement within training examples directly reduces the amount of data required to achieve a desired level of accuracy. This theoretical finding challenges the data-hungry nature of classical learning systems and opens a pathway toward data-efficient quantum learning paradigms.

Quantum discord represents a weaker but more robust form of quantum correlation that persists even in mixed or noisy states. The third study shows that quantum discord can drive useful computation in the Deterministic Quantum Computing with One Qubit (DQC1) framework, where only a single pure qubit interacts with a mixed-state register. This setup is particularly suited for today’s NISQ devices because it avoids the need for large-scale entanglement and complex multi-qubit measurements.  
The experimental results confirm that discord-based kernels can achieve meaningful classification accuracy while maintaining resilience against decoherence and readout noise. These findings position quantum discord as a practical computational resource for early-stage quantum machine learning, capable of enabling near-term quantum advantage before full-scale fault-tolerant quantum computers become available.

Future research in quantum machine learning should progress along a staged but interdependent path that moves from theory to data representation, to empirical validation, to system integration. First, the field needs a more realistic theoretical foundation: generalize quantum NFL results to imperfect optimization, label noise, and device noise, replacing the idealized assumption of perfect training conditions. The generalize quantum NFL should predict how effective sample complexity scales with the product of dataset size and usable correlation after noise, rather than with either factor alone.

Building on that foundation, research should develop scalable encodings that map continuous, high-dimensional data to quantum states while preserving useful correlations under hardware constraints. Comparative studies of angle/phase, amplitude, and hybrid encodings need to quantify trade-offs among qubits, depth, retained quantum correlation, and noise sensitivity both synthetic and real-world large-scale continuous data. With theory and encodings in place, the next priority is empirical identification or construction of entangled dataset and the demonstration that they reduce sample complexity in line with theory. This requires benchmarks with tunable Schmidt rank. lthough entangled datasets can be mathematically simulated or experimentally engineered, measuring entanglement in naturally occurring data—such as electroencephalogram (EEG) signals, sensory data, or high-dimensional multimodal inputs—will have a significant impact on the QML field. We need to develop quantitative tools to detect “quantum-like” correlations in classical data, where quantum-correlation measures can further be utilized as feature selectors for classical ML algorithms and can also lead to the development of a new family of quantum machine learning algorithms.

To make results comparable and hardware-relevant, evaluation must move beyond *p-values* to a quantum-native metric suite that jointly reports learning performance (risk, margins, kernel conditioning), physical indicators (state/process fidelity, coherence consumed, discord generated, depth, two-qubit gate count), robustness (accuracy under injected noise and compilation changes), and efficiency (shots, wall-clock time, qubit-seconds, classical post-processing). A reproducible leaderboard that aggregates these metrics would enable fair comparisons across devices and methods.

Finally, integration should focus on hybrid architectures that deploy entanglement where it buys expressivity and discord where it buys robustness. Two-tier pipelines—using shallow entanglement to create expressive features and a DQC1-style kernel estimator for scalable evaluation, followed by a classical learner—can be paired with explicit “coherence budgeting,” in which the purity and feature-map depth control accuracy–cost trade-offs predicted by the theory. Program-level risks such as noise and encoding bottlenecks can be mitigated with error-mitigation techniques, kernel regularization or projection, and classical dimensionality reduction prior to quantum mapping. Taken together, this roadmap closes the loop from theory to deployment: realistic bounds specify what is learnable under noise; encodings make real data compatible with those bounds; experiments validate that quantum correlations reduce sample needs; metrics quantify true advantage; and hybrid systems translate entanglement’s expressivity and discord’s resilience into scalable, hardware-relevant quantum machine learning system.

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