Estimation of Lyapunov Spectra from a Time Series

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Abstract

Nonlinear dynamical systems can be identified by their sensitivity to initial conditions. This is captured by a set of invariant measures called Lyapunov exponents. For accurate esitmation of these exponents from an observed time series, we need knowledge of the trajectory representing time evolution of the system attractor. In this paper we present an explanation of reconstructed phase spaces. We then build on the motivation behind Lyapunov exponents and present an algorithm for their estimation. Experiments were performed on two standard chaotic series and on a deterministic, periodic series. Results indicate that positive and zero exponents can be estimated reliably even in the presence of noise.

1. Introduction

A time series representing an observable of a nonlinear dynamical system contains information that is hard to capture using conventional tools such as Fourier analysis. A regular Fourier spectrum provides useful information if the signal is generated by a linear source. However, for signals generated by a nonlinear source, the Fourier spectrum will typically reveal a wideband (infinite dimensional) structure, even though the dynamical system exists in a finite dimensional space [1]. Hence, an alternative characterization of such time series is desired.

A nonlinear system can be described using a state-space model with a number of observable output states. The time evolution of these observables in the state-space constitutes a trajectory. Lyapunov exponents associated with a trajectory provide us with a measure of average rates of convergence and divergence of surrounding trajectories. These are considered to be an important invariant characterization of the underlying dynamical system. Lyapunov exponents are also a good measure to distinguish between fixed points, periodic, quasi-periodic and chaotic motions [2].

The outline of the paper is as follows. In Section 2, we explain concepts of reconstructed phase spaces and embedding. We also illustrate these concepts with an example of a time series generated from a Lorentz system. In Section 3, we introduce the concept of Lyapunov exponents and illustrate the algorithmic implementation of Lyapunov spectra estimation from an observed time series. In Section 4, we present the experimental setup and the simulation results exemplifying the theory presented.

2. Reconstructed Phase-Space

Computation of the Lyapunov exponents presupposes that we have full knowledge of the dynamics of a system. This requires us to have measurements for each possible variable in the system. However, in practice, we usually have only one time series measurement. In such cases, though we cannot find the exact phase-space of the system, a pseudo phase-space (equivalent to the original phase-space in terms of the system invariants) may still be constructed. This pseudo phase-space [2][3] is called the Reconstructed Phase-Space (RPS).

To form the RPS matrix from a time series, we need to know the inherent system dimension, d. From knowledge of the system dimension, an upper bound on the dimension of the RPS can be placed. Taken's theorem [2] states that we can construct an RPS that is equivalent to the original phase-space by embedding with a dimension $m \geq 2d+1$. Though this theorem provides us with a theoretically sufficient bound, such a bound is not necessary in practice. Most systems can be embedded in much lower-dimensional spaces. There are two methods [2] by which embedding can be achieved: Time Delay and Singular Value Decomposition (SVD).

2.1. Time Delay Embedding

The simplest method to embed scalar data is the method of delays. This works by reconstructing the pseudo phase-space from a scalar time series, by using delayed copies of the original time series as components of the RPS. It involves sliding a window of length *m* through the data to form a series of vectors, stacked row-wise in the matrix. Each row of this matrix is a point in the reconstructed phase-space.

Letting $\{x_i\}$ represent the time series, the RPS matrix is represented as given by equation 1, where *m* is the embedding dimension and τ is the embedding delay (in samples).

$$X = \begin{pmatrix} x_0 & x_\tau & \cdots & x_{(m-1)\tau} \\ x_1 & x_{1+\tau} & \cdots & x_{1+(m-1)\tau} \\ x_2 & x_{2+\tau} & \cdots & x_{2+(m-1)\tau} \\ \vdots & & \vdots & \end{pmatrix} \quad (1)$$

Fixing an optimal value of *m* requires domain specific knowledge about the time series being analyzed. The method of false-nearest neighbors can be useful to some extent in this regard. The delay parameter, τ , is chosen such that the structure of the original attractor is captured in the RPS. Underestimating the value for delay leads to highly correlated vector elements, which would now be concentrated around the diagonal in the embedding space, and the structure perpendicular to the diagonal is not captured adequately. On the other hand, a very large estimate of the delay will result in the elements of each vector to behave as if they are randomly distributed. Quantitative tools like auto-correlation and auto-mutual information are useful guides in choosing the optimal value of τ .

2.2. SVD Embedding

Time delay embedding requires knowledge of the optimal delay for reconstructing the RPS. Though this parameter can be estimated using plots of auto-correlation or auto-mutual information, this value is not guaranteed to yield the correct RPS. Moreover, if the time series is corrupted by noise, time delay embedding can yield RPS matrices that are very poor representations of the actual phase-space. For these reasons, another method of embedding using SVD is preferred.

SVD embedding works in two stages. In the first stage, an initial RPS matrix is formed from the time series using time delay embedding with delay of one sample. The dimension for this embedding is larger than the actual embedding dimension and is referred to as SVD window size. The second stage proceeds by reducing this matrix, using SVD, to a matrix with the same number of rows but a number of columns equal to the embedding dimension.

The application of SVD in reducing the noise level is well known in signal processing. It is expected that the Lyapunov exponents calculated from SVD embedded RPS will be more robust to noise than those estimated from time delay embedding. For this reason, we restrict our analysis to SVD embedding.

2.3. Simulation Using the Lorentz System

A popular dynamical system that is used in the study of chaos is the Lorentz system [3] of differential equations:

$$\begin{aligned} \dot{X} &= \sigma(Y - X) \\ \dot{Y} &= rX - Y - XZ \\ \dot{Z} &= -bZ + XY \end{aligned} \quad (2)$$

Figure 2 shows the attractor structure of the system with parameters $\sigma = 16.0$, $r = 40.0$ and $b = 4.0$ in the x-y plane. It also shows the attractor structure in the reconstructed phase space generated by SVD embedding ($m = 3$ and SVD window size=15). A visual inspection of this figure reveals that the original structure of the system attractor is preserved in the RPS.

3. Lyapunov Exponents

The analysis of separation in time of two trajectories with infinitely close initial points is very important in the analysis of nonlinear dynamical systems [1]. For a system whose evolution function is defined by a function *f*, we need to analyze

$$\Delta x(t) \approx \Delta x(0) \frac{d}{dx} (f^N) x(0) \quad (3)$$

To quantify this separation, we assume that the rate of growth (or decay) of the separation between the trajectories is exponential in time. Hence we define the exponents, λ_i as

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$$\lambda_i = \lim_{n \to \infty} \frac{1}{n} \ln(\mathrm{eig} \prod_{\mathrm{p}=0}^{\mathrm{n}} \mathrm{J(p)}) \quad (4)$$

where, J is the Jacobian of the system as the point p moves around the attractor. These exponents are invariant characteristics of the system and are called Lyapunov

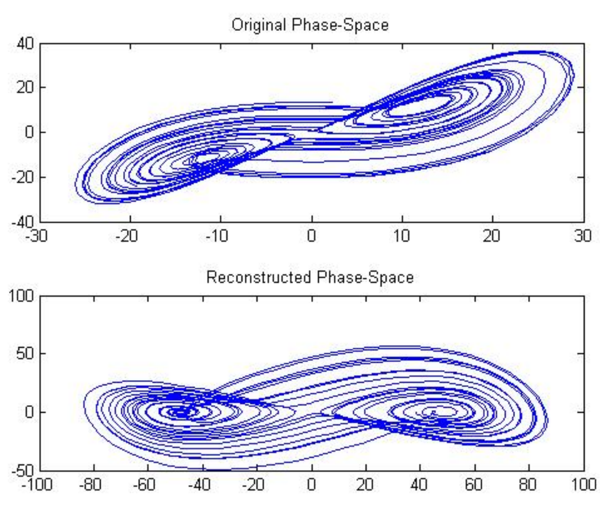

Figure 2. The attractor structure of the Lorentz system in the original and reconstructed state-space.

exponents.

For a multidimensional system, we have as many Lyapunov exponents as the dimension of the system. They may be zero, negative or positive. For a dynamical system with a bounded attractor, the sum of all Lyapunov exponents should be less than or equal to zero. Zero exponents indicate that the system is a flow, while the positive ones indicate that the system is chaotic. Negative exponents characterize a system's tendency to pull an evolving trajectory towards the basin of attraction.

3.1. Computation of Lyapunov Exponents

An algorithm to compute Lyapunov exponents is given in Figure 3. We begin by embedding the input time series to provide the RPS matrix with each row representing a point on the trajectory. With the first point as center, we form a neighborhood matrix, each row of which is obtained by subtracting a neighbor from the center. Next, we find the evolution of each neighbor and form the evolved neighborhood matrix. The trajectory matrix is computed by multiplying the pseudo-inverse of neighborhood matrix with the evolved neighborhood matrix. The Lyapunov exponents are calculated from the eigen-values of the trajectory matrix. These exponents are averaged by evolving the center point through the trajectory. Since direct averaging has numerical problems, an iterative QR decomposition method (Treppen iteration) is used.

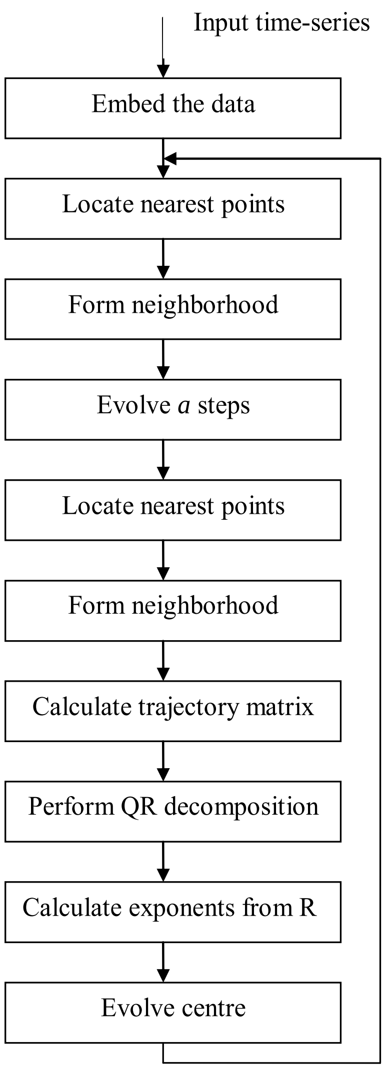

Figure 3. An algorithm to compute Lyapunov spectra from a scalar time series [2]

4. Experimental Setup and Results

To demonstrate the accuracy of our implementation, two chaotic systems - Lorentz and Rossler, are considered. The equations for Lorentz system are given in equation 2 and the parameters considered for that system are $\sigma = 16.0$, $r = 40.0$ and $b = 4.0$. The parameters for the Rossler system [4], [5] are a = 0.15, b = 0.2, c = 10. The accepted dimensionality of both the systems is three and the Lyapunov exponents for the Lorentz and Rossler systems with the given parameters can be calculated numerically and were found to be (+1.37, 0, -22.37) and (0.090, 0.00, -9.8), respectively.

To gain confidence about the accuracy of our algorithmic implementation, we also tested it on a sinusoidal signal of frequency 1 Hz sampled using a sampling interval of 0.06s. Since a sinusoidal signal generates a periodic attractor (a circle), it is expected to have zero and negative exponents only. The absence of positive exponents indicates that the trajectory is stable and never diverges during its evolution. The presence of a negative exponent indicates the tendency of the system attractor to pull any trajectory divergence towards the basin of attraction. Previously reported exponents [6] for this sinusoid are (0.00, 0.00, -1.85).

Figures 3-5 illustrate Lyapunov spectra estimates of various time series (with and without noise) as a function of two parameters – SVD window size and number of nearest neighbors. The SNR for the noisy series was set to 10dB. When varying SVD window size, the number of neighbors was fixed at 15 for the clean series and at 50 for the noisy series. When varying number of neighbors, the SVD window size was fixed at 15 for the clean series and at 50 for the noisy series. In all cases, the evolution step was fixed at 8 samples.

As can be seen from the figures, estimates of the positive and zero exponents from the clean series converge to the expected values. Also note that estimates of these exponents for noisy series converge for an SVD window size greater than 60 samples and for number of nearest neighbors greater than 50. However, the variation of the negative exponents in all cases does not follow any trend. This does not harm most nonlinear analyses because only positive exponents are used for the characterization of chaos.

5. Conclusions

In this paper, we provided an explanation and motivation for reconstructed phase spaces using the methods of time delay and SVD embedding. We then explained Lyapunov exponents as a dynamical invariant measure and presented an algorithm for estimating them from an observed time series. We tested our implementation on two standard chaotic series and on a deterministic, periodic series. Results indicating accurate estimation of positive and zero exponents were documented. It is hoped this algorithm will provide us with a useful signal processing tool to perform signal characterization and system identification in nonlinear settings.

6. Acknowledgements

This material is based upon work supported by the National Science Foundation under Grant No. IIS-0414450. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the National Science Foundation.

7. References

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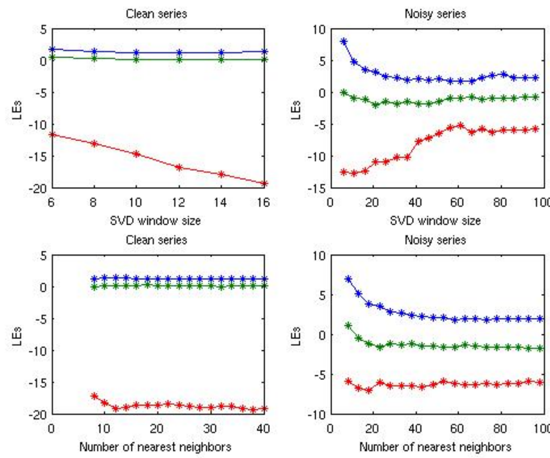

Figure 4. Lyapunov exponents from a time series generated from a Lorentz system

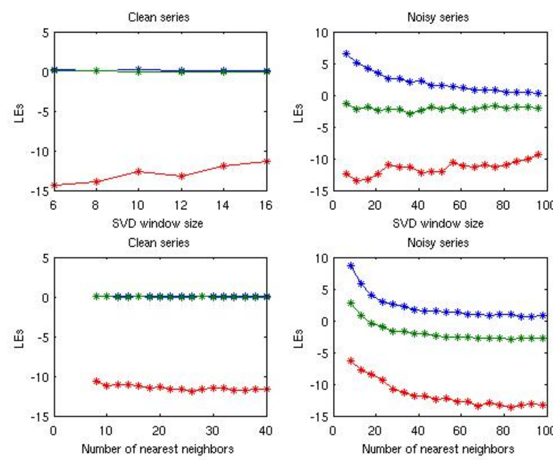

Figure 5. Lyapunov exponents from a time series generated from a Rossler system

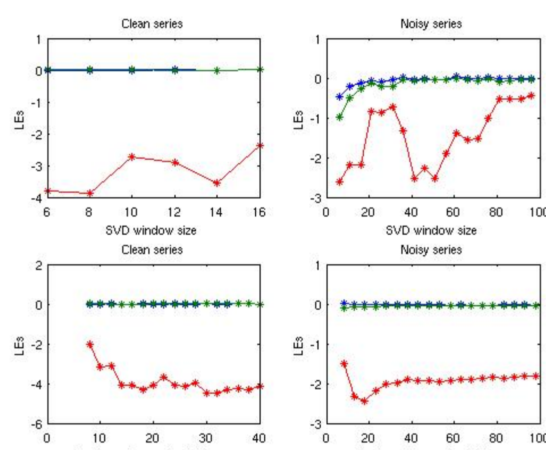

Figure 7. Lyapunov exponents from a sinusoidal signal with and without noise