Exam 1

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1 Problem 1

1.1 a

In this problem we are given two probability distributions denoted as $P(X|\omega_1)$ and $P(X|\omega_2)$, both distributions are uniform over a certain range, however only $P(X|\omega_2)$ has a fixed **location**, in other words $P(X|\omega_1)$ is not fully known to us. This could mean that we are receiving data from two classes, where one we know its mean and the other has an unknown mean. In this situation, we find the probability of error for every location $P(X|\omega_1)$ can have its mean located.

$$P(X|\omega_1) = \begin{cases} 1 : \alpha - 0.5 <= x <= \alpha + 0.5 \\ 0 : elsewhere \end{cases} \tag{1}$$

$$P(X|\omega_2) = \begin{cases} 1 : -0.5 <= x <= 0.5 \\ 0 : elsewhere \end{cases} \tag{2}$$

$$P(error, \omega_2) = \int_{R_2} P(X|\omega_1) \cdot P(\omega_1) dx \tag{3}$$

The probability of error $p(error)$ is also dependent on the classes' prior $P(\omega_1)$ and $P(\omega_2)$ and of course the variable $alpha$ shown in Eq 3. To find this probability of error, α is varied and the probability of error is computed as:

The probability of error for a varying $alpha$ is computed similar to the convolutional integral, the arrow (*sorry I know it is small*) shows the shift done to the second distribution resulting the probability of error for a varying value of α. Finally, the prior probabilities are also varied as illustrated in Figure 3.

1.2 Part b

Part b of the problem examines the probability of error for two Gaussian instead of the uniform distributions. This case clearly different, and $alpha$ will now be represented as the mean of the distribution. Hence, $P(X|\omega_1)$ will have a fix variance, but its mean will vary. The probability of error will then be calculated as the convolutional integral again. The results for such case are illustrated in Figure 4.
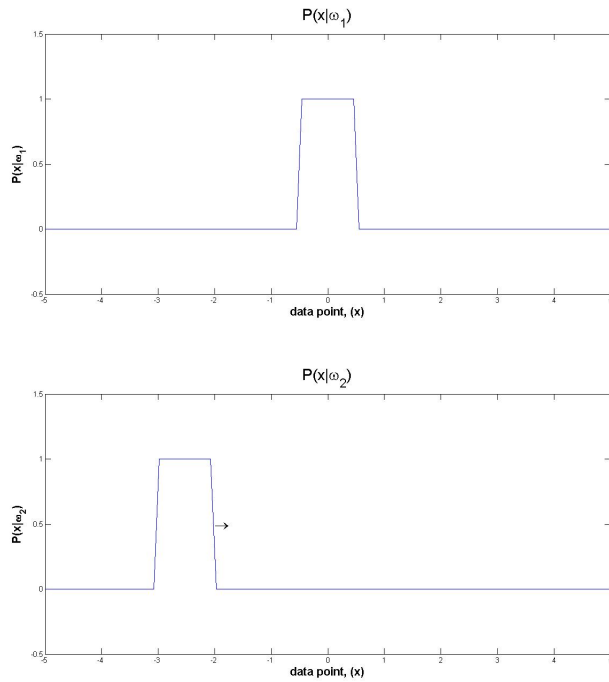
Figure 1: Probability distribution for class $\omega_2 and \omega_2$ (R1 and R2)

2 Problem 2

In this problem, an expression for the maximum likelihood estimate is derived for a uniform density denoted as $p(x|\theta)$. The approach to the derivation is listed in the equation in Eq 5 where $px|\theta$ is defined as:

$$P(X|\theta) = \begin{cases} \dfrac{1}{\theta} : 0 \le x \le \theta \\ 0 : elsewhere \end{cases} \tag{4}$$

Since $P(D|\theta)$ is the likelihood of θ for a given set of data points, the maximum likelihood will be the value that maximizes the $P(D|\theta)$.

$$\begin{aligned} P(D|\theta) &= \Pi p(x_k|\theta) \\ &= \prod_{k=1}^{n} \frac{1}{\theta}, \text{ if } (0 \le x \le \theta) \\ &= \frac{1}{\theta^n} \end{aligned} \tag{5}$$

In Eq 5 $P(D|\theta)$ is exponentially decreasing as the number of points are created or drawn, therefore the maximum value has to be when n = 0. The way that I am interpreting this, is if we have a uniform distribution, the most likely to occur at the end of that distribution.
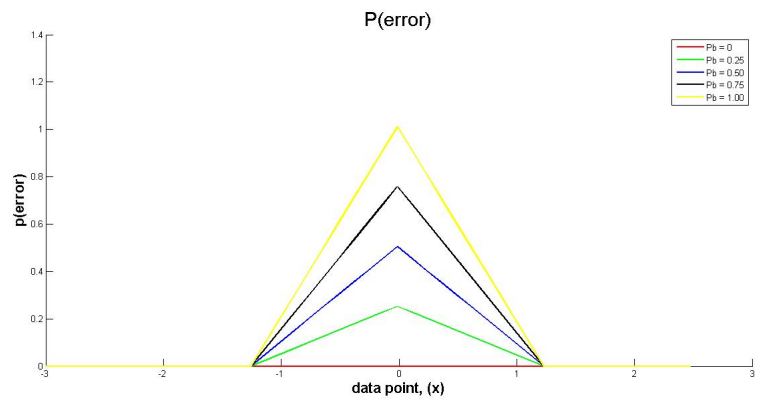
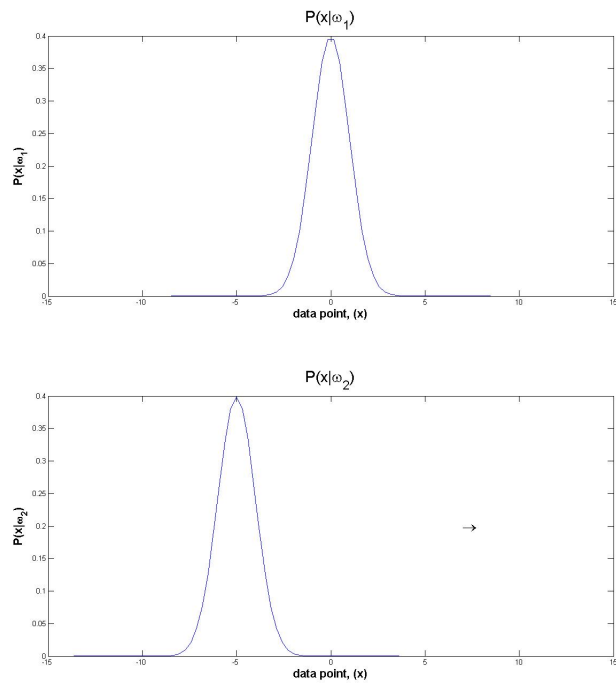Figure 2: Probability of error as a function of α and with differ $P(\omega)$

3

P(x|ω₁)

P(x|ω₂)

Figure 3: Gaussian distribution for class $\omega_2 and \omega_2$ (R1 and R2)
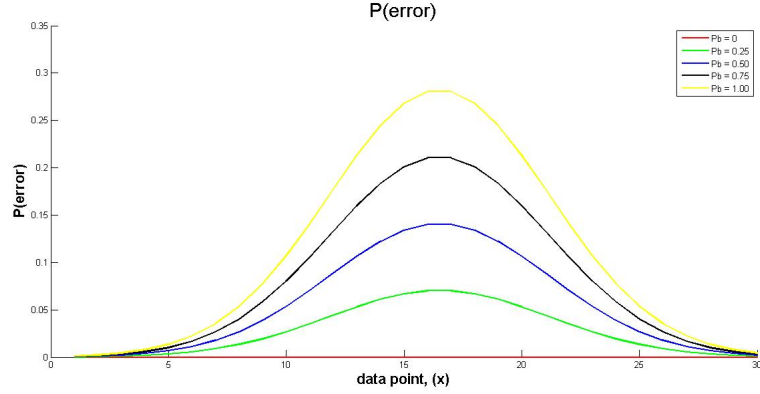
Figure 4: Probability of error as a function of α and with differ $P(\omega)$, for two Gaussian

3   Problem 3

3.1   Analysis

In Problem 3, the value of a filter α coefficient is estimated. The input to the filter is a White Gaussian Noise, the assumption in the problem is that the output of the filter is also a Gaussian signal, therefore, this problem can be solved using a MLE. The filter's Z-transfer function is described in Eq 6, from the Z-transfer function can be obtained the difference equation which is shown in Eq 7.

$$\frac{1}{1 - \alpha z^{-1}} \tag{6}$$

$$y[n] = x[n] + \alpha \cdot y[n-1]; \tag{7}$$

An estimate of the filter coefficient can be obtained the the Mean Square Error (MSE) which can be derived through the following steps.

$$
\begin{aligned}
MSE(y[n], \alpha y[n-1]) &= \\
&= \sum_{k=1}^{n} \left\{ y[n] + \alpha y[n-1] \right\} \\
&= \sum_{k=1}^{n} \left\{ y[n]^2 + \alpha^2 y[n-1]^2 + 2y[n]\alpha y[n-1] \right\}
\end{aligned} \tag{8}
$$

To solve for α we take derivative of the MSE is set it equal to zero resulting in the following:

$$\frac{\partial MSE}{\partial \alpha}$$

Hence, (9)

$$\alpha = \frac{-\sum_{k=1}^{n} y[n]y[n-1]}{\sum_{k=1}^{n} y[n-1]y[n-1]}$$

This MSE shows that α can be found only using the outputs. On the other hand, if we carry our assumption of the data being Gaussian, we may proceed as follow:

$$y[n] \sim N(\mu, \sigma^2)$$

$$\sim \frac{1}{\sqrt{2\pi\sigma^2}} \cdot exp\left(\frac{-y[n]-\mu}{\sqrt{2\sigma^2}}\right)^2 \qquad (10)$$

The expression for μ can be replaced by $\mu = E(y[n]) = E(-\alpha y[n-1])$, therefore $y[n]$ can be shown as follow:

$$y[n] \sim \frac{1}{\sqrt{2\pi\sigma^2}} \cdot exp\left(\frac{-y[n]+\alpha y[n-1])}{\sqrt{2\sigma^2}}\right)^2 \qquad (11)$$

To simply it the natural logarithm is taken and similarly, the derivative is calculated and set to zero to find that the filter coefficient equals to:

$$\alpha = -\frac{\sum_{k=1}^{n} y[n]y[n-1]}{\sum_{k=1}^{n} y[n-1]y[n-1]} \qquad (12)$$

3.2   Simulation

The estimator can be simulated in Matlab creating a Gaussian variable and filtering through filter in Eq 7. This is shown in Figure 5, where $\alpha = 0.5$
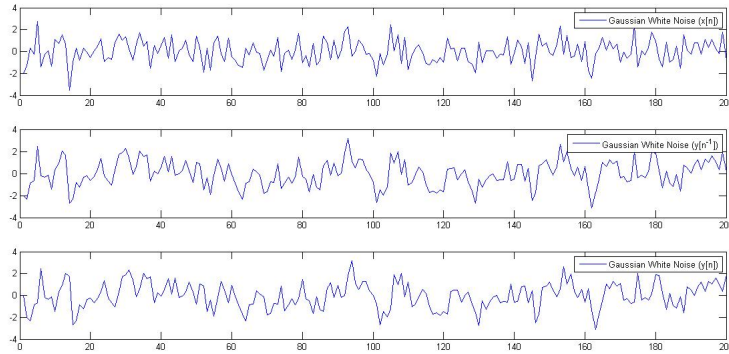

Figure 5: Gaussian variable x[n] and the filter's out put y[n-1] and y[n]

Using the implementation Eq 12, the filter coefficient is calculated to be 0.5202 for 200 data points.