Homework 3

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1 Problem 1

The first problem of assignment 3 consisted in finding the variance and the mean of a Gaussian random variable. By varying the length of the GRV. Therefore, the using the Maximum Likelihood Estimates, the mean and variance will be found for a Gaussian Random Variable with varying length. Using Matlab, the MLE is quickly obtained and yields to the following results:

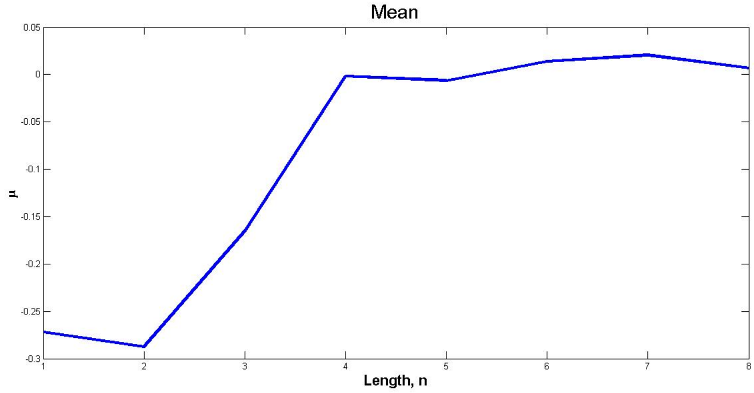

Figure 1: Mean estimated using the MLE

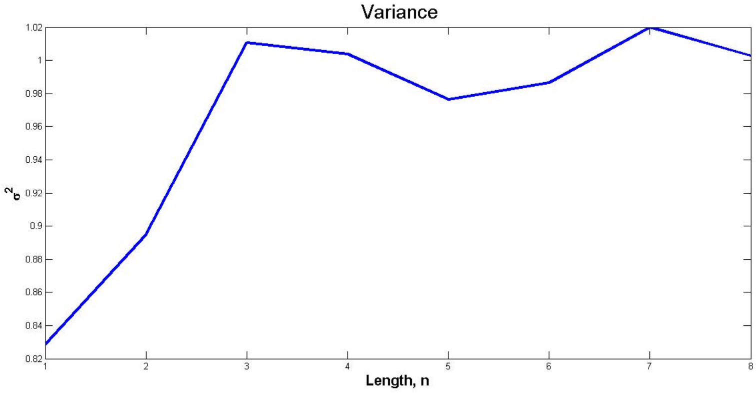

Figure 2: Variance estimated using the MLE

1.1 Obsevation

2 Problem 2

2.1 White Gaussian Noise

In the second problem, the autocovariance and autocorrelation of a white Gaussian noise were observed. The two are very important properties of signal since there may be signals or processes that cannot be described with a simple function because of their discontinuities. The autocorellation of a signal is a function that describes how similar a sample is to another sample of the same signal at a different time. This is very important in several areas of other than engineering. For instance, it is almost always possible to find some kind of coherence within a track of music. Modern music usually repeat a chorus or verse so there must have a high degree coherence or autocorellation. The autocorrelation function takes that idea of coherence and assigns a value that describes the coherence or correlation. In a signal such as white Gaussian noise which is very discontinuous the autocorrelation is used to describe this signal. The autocorollation is defined as:

$$R_{XX}\left(t, t+\tau\right) = E\left[X\left(t\right) \cdot X\left(t+\tau\right)\right] \tag{1}$$

The sample points of the white Gaussian noise are from a Gaussian distribution with mean(μ) equal to zero and variance (σ^2) equal to one. The white Gaussian noise being very non-linear, its autocorrelation function is found analytically using Matlab. By using the function *wgn* to create the white Gaussian noise, the autocorellation function for 1000 samples was found. For a vector of 1000 numbers, the autocorellation is found by shifting the vector by a sample and computing the dot product between the new shifted vector $X(n+k)$ and the original vector $X(n)$. This process was accomplished using the snippet illustrated below:

```
y_left = zeros(n);
w = rotator(x,n);
for j = 1:n
    y_left = MrShifter(w,n,j);
     R_left = dot(x,y_left);
end

for j = 1:n
    y = MrShifter(x,n,j);
    R = dot(x,y);
end
```

Snippet 1: Finding the autocorellation function of the white Gaussian noise. The function *MrShifter* is used to shift the the samples of the Gaussian noise

Using Snippet 1, the autocorellation function was found and is illustrated in Figure 3 below:

$$R_{XX}\left(t, t+\tau\right) = n \cdot \delta(\tau) \tag{2}$$

Similarly, the function of the autocovariance function can be found by executing a similar process. Given an $1 \times n$ matrix, the autocovariance $C_{(i,j)}$ is defined as a matrix:

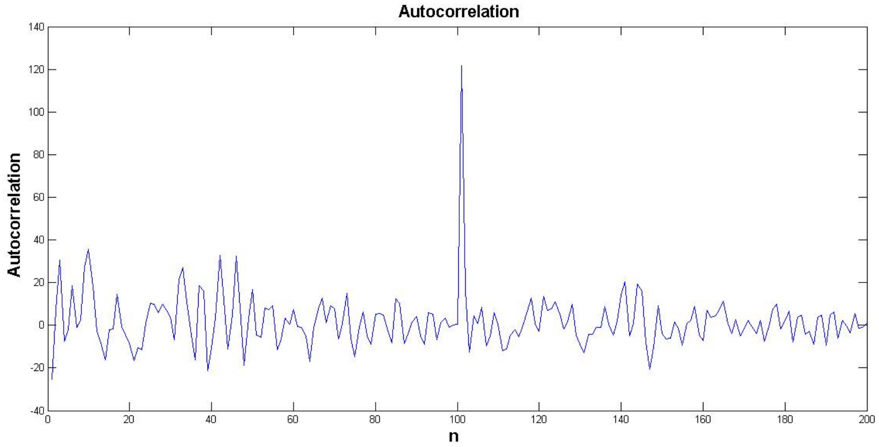

Figure 3: Autocorellation function of a white Gaussian noise

$$C_{XX} = \begin{pmatrix} E\left[X\left(t_1\right)\cdot X\left(t_1\right)\right] & E\left[X\left(t_1\right)\cdot X\left(t_2\right)\right] & \cdots & E\left[X\left(t_1\right)\cdot X\left(t_n\right)\right] \\ E\left[X\left(t_2\right)\cdot X\left(t_1\right)\right] & E\left[X\left(t_2\right)\cdot X\left(t_2\right)\right] & \cdots & E\left[X\left(t_2\right)\cdot X\left(t_n\right)\right] \\ \vdots & \vdots & \ddots & \vdots \\ E\left[X\left(t_n\right)\cdot X\left(t_1\right)\right] & E\left[X\left(t_n\right)\cdot X\left(t_2\right)\right] & \cdots & E\left[X\left(t_n\right)\cdot X\left(t_n\right)\right] \end{pmatrix} \tag{3}$$

With a 1000 datapoints, (n = 1000), a 1000 × 1000 matrix is expected to result from the process. This matrix may be huge and inconvenient to illustrate, however the properties of the white Gaussian noise result in a matrix where the diagonal numbers represent the variance σ_x^2 of the Gaussian distribution used to generate the noise. Hence the result can be found to be the following matrix:

$$C_{i,j} = \begin{pmatrix} \simeq \sigma_x^2 & \simeq 0 & \cdots & \simeq 0 \\ \simeq 0 & \simeq \sigma_x^2 & \cdots & \simeq 0 \\ \vdots & \vdots & \ddots & \vdots \\ \simeq 0 & \simeq 0 & \cdots & \simeq \sigma_x^2 \end{pmatrix} \times n \tag{4}$$

2.2 Observation

It is very common to analyze a signal in the time domain, however, in those specific process, where there are non-linear, those processes cannot be expressed in the frequency domain. The autocorrelation function consist of properties that enable the signal to be described in the frequency domain since R_{XX} is bounded, where in certain cases the random processes the Fourier Transform may not exist. Therefore the Power Spectral Density is described as the Fourier

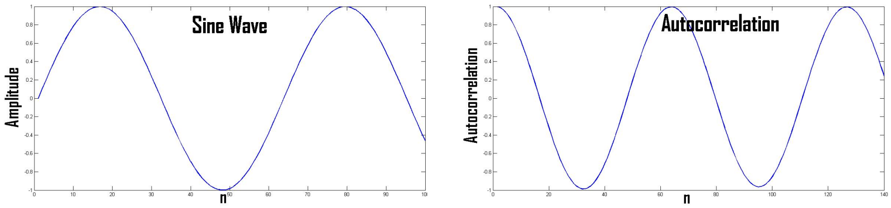

Figure 4: Sine function (left) and auto-correlation function (right)

Transform of the auto-correlation:

$$F\left(R_{XX}\left(\tau\right)\right) \tag{5}$$

2.3 Sine

In a similar manner, a sinusoidal wave can be observed through its resulting auto-correlation function. The sinusoidal denoted as $X[n] = sin[\frac{2\pi\cdot\cdot n}{T_s}]$. With this, the auto-correlation function can be calculated as follow:

$$X[n] = A \cdot sin\left(\frac{2\pi \cdot n}{T_s}\right) \tag{6}$$

$$R_{XX}(k) = E\left\{A \cdot sin\left(\frac{2\pi \cdot n}{T_s}\right) \cdot A \cdot sin\left(\frac{2\pi \cdot n - k \cdot 2\pi}{T_s}\right)\right\} \tag{7}$$

$$R_{XX}(k) = \frac{A^2}{2} \cdot cos\left(\frac{2\pi \cdot k \cdot n}{T_s}\right) \tag{8}$$

However, in simulation, the auto-correlation function is scaled by the by half the number of samples that make up the sinusoidal wave.

$$scalor = \frac{n}{2} \tag{9}$$

In Matlab, the auto-correlation between the function is implemented using the *'MrShifter'* function whose purpose is to shift the vector of sampled data to the left and adding new samples to the tail of the vector. This shift is analogous to delaying a sinusoidal in time. These results can be observed in Figure 4, where the auto-correlation function of the sinusoidal wave are illustrated.

2.4 Observation

Observing the auto-correlation function, a relation ship can between the second moment of the mean and the auto-correlation with k = 0:

$$\mu^2 = E\left\{sin\left(x\right)\right\}^2 \tag{10}$$

,

$$\begin{aligned} R_{xx}(0) &= E\left\{sin\left(x\right)\cdot sin\left(x - k|_{k=0}\right)\right\} \\ &= E\left\{sin\left(x\right)\right\}^2 \end{aligned} \tag{11}$$

Hence, it can be said that,

$$\begin{aligned} R_{xx}(0) &= \mu^2 \\ &= E\left\{sin\left(x\right)\right\}^2 \\ &= \frac{A^2}{2} \end{aligned} \tag{12}$$

We can also observe that a relationship between the correlation coefficients at different delays k. This relationship is seen in:

$$\begin{aligned} R_{xx}(0) &= \left|R_{xx}(k)\right| \\ &= \frac{A^2}{2} \end{aligned} \tag{13}$$

Hence, the auto-correlation function of a sinusoidal wave is equal to:

$$\begin{aligned} R_{xx}(k) &= \left|R_{xx}(k)\right|\cdot cos(k) \\ &= \left|R_0(k)\right|\cdot cos(k) \\ &= \frac{A^2}{2}\cdot cos(k) \end{aligned} \tag{14}$$

In terms of the covariance of the sinusoidal waveform, it is computed in a similar manner as the covariance of the Gaussian random variable. The covariance matrix of the sinusoidal was found to be a 2 × 2 matrix.

$$C_{XX} = \begin{pmatrix} \frac{A^2}{2}cos(0) & \frac{A^2}{2}cos(\{k+1\}) & \cdots & \frac{A^2}{2}cos(\{k+n-1\}) \\ \frac{A^2}{2}cos(\{k+1\}) & \frac{A^2}{2}cos(0) & \cdots & \frac{A^2}{2}cos(\{k+n\}) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{A^2}{2}cos(\{k+n-1\}) & \cdots & \cdots & \frac{A^2}{2}cos(0) \end{pmatrix} \tag{15}$$

$$C_{XX} = \begin{pmatrix} cos(0) & cos(\{k+1\}\cdot pi) & \cdots & cos(\{k+n-1\}\cdot pi) \\ cos(\{k+1\}\cdot pi) & cos(0) & \cdots & cos(\{k+n\}\cdot pi) \\ \vdots & \vdots & \ddots & \vdots \\ cos(\{k+n-1\}\cdot pi) & \cdots & \cdots & cos(0) \end{pmatrix} \times \frac{A^2}{2} \tag{16}$$

The rank of this matrix can be computed doing a Reduced Row Echelon form which will reduce the matrix only its pivot elements. After reduced Row Echelon form, this matrix will only have one pivot, this *not 100 percent sure* results from the phase of the signal. Because the phase of the signal is repeated, the sinusoidal has one element in common. This is also why the autocorrelation is oscillating.

3 Problem 3

3.1 Part 1

At this stage, the previously created Gaussian noise will be processed through an IIR filter with transform $H[z]$. The filter will have one unknown coefficient α. However, this unknown coefficient can be found using digital signal processing techniques that will find the value of α as the number of data is increasing. The z-transform of the filter is described as:

$$H[z] = \frac{1}{1 - 0.75z^{-1}} \tag{17}$$

To theoretically find the variable α, the z-transform be will altered to its difference equation equal to:

$$y[n] = x[n] - \alpha \cdot y[n-1] \tag{18}$$

To obtain the unknown variable, the output signal can be delayed by one sample, this sample will be $y[n-1]$. An assumption must be made to find the value of α, the output $y[n]$ will be assumed to be Gaussian distributed, which will allow the Maximum Likelihood to be used for computing the unknown variable or simply the Mean Square Error. Using the Mean Square Error (MSE):

$$\begin{aligned} MSE(y[n] + \alpha \cdot y[n-1]) &= \sum_{k=1}^{n} \big(y[n] - \alpha \cdot y[n-1]\big)^2 \\ &= \sum_{k=1}^{n} \Big(y[n] \cdot y[n] + \alpha^2 \cdot 2y[n-1] \cdot y[n-1] + 2\alpha \cdot y[n] \cdot y[n-1]\Big) \end{aligned} \tag{19}$$

The derivative of the MSE is taken with respect α and leads to solving an expression for the value of α.

$$\frac{\partial MSE}{\partial \alpha} = \sum_{k=1}^{n} \big(2\alpha \cdot y[n] \cdot y[n-1] + 2y[n-1] \cdot y[n-1]\big) \tag{20}$$

$$0 = \sum_{k=1}^{n} \big(2\alpha \cdot y[n-1] \cdot y[n-1] + 2y[n] \cdot y[n-1]\big) \tag{21}$$

$$\sum_{k=1}^{n} \big(2\alpha \cdot y[n-1] \cdot y[n-1]\big) = \sum_{k=1}^{n} \big(-2y[n-1] \cdot y[n]\big) \tag{22}$$

$$\alpha = -\frac{\sum_{k=1}^{n} \big(y[n] \cdot y[n-1]\big)}{\sum_{k=1}^{n} \big(\cdot y[n-1] \cdot y[n-1])\big)} \tag{23}$$

n	$alpha_{filter}$	$alpha_{estimated}$
500	0.75	0.7961
1000	0.75	0.7039
1500	0.75	0.7238
2000	0.75	0.7504
2500	0.75	0.7330
3000	0.75	0.7391
3500	0.75	0.7539
4000	0.75	0.7427
4500	0.75	0.7430
5000	0.75	0.7520

Table 1: Results obtained for α for various values of n

In Matlab, this can easily be computed as the ratio of the dot products between the previous samples and the current samples as illustrated in Equation 24. The trend depicted in Table 1 shows, that a high number of sample assures an increase in precision in terms of finding α

3.2 Part 2

A similar expression of an IIR filter can be explored in the quest of an unknown filter coefficient α, the z-transform is described by equation 25:

$$H[z] = \frac{1}{1 + 0.25 \cdot z^{-1} - \alpha \cdot z^{-2}} \tag{24}$$

And the difference equation comes out as:

$$y[n] = x[n] - 0.25 \cdot y[n-1] - \alpha \cdot y[n-2] \tag{25}$$

This is very similar to the previous Part, (Part 1) by delaying the vector of samples by the same process can be repeated. Using an MLE instead of a MSE, an assumption about the result of the data $y[n]$ must be made, the output is Gaussian. From there it can be said that $y[n] \sim N(\mu, \sigma^2)$.

$$y[n] \sim \sqrt{\frac{1}{2\pi\sigma^2}} \cdot exp\left(\frac{1}{2\sigma^2} \cdot \left(y[n] - \mu^2\right)\right) \tag{26}$$

Applying the natural log (ln) unto the expression show in 27. Then the partial derivative in terms of α is calculated to obtain the expression of:

$$\alpha = -\frac{\sum_{k=1}^{n}\left(y[n-2]\right)\sum_{k=1}^{n}\left(y[n] - 0.25 \cdot y[n-1]\right)}{\sum_{k=1}^{n}\left(y[n-2]\right) \cdot \sum_{k=1}^{n}\left(y[n-2]\right)} \tag{27}$$

n	$alpha_{filter}$	$alpha_{estimated}$
500	0.75	-0.7599
1000	0.75	-0.7813
1500	0.75	-0.7546
2000	0.75	-0.7477
2500	0.75	-0.7531
3000	0.75	-0.7542
3500	0.75	-0.7600
4000	0.75	-0.7320
4500	0.75	-0.7358
5000	0.75	-0.7494
5500	0.75	-0.7547
6000	0.75	-0.7530
10000	0.75	-0.7546

Table 2: Results obtained for α for various values of n

3.3 Observation

4 Problem 4

Problem 4 looks at the same concept as question 3, an IIR filter is observed as its coefficient α is varied through time as a function of a sinusoidal waveform with 1 Hz frequency.

$$\alpha = 0.75 + 0.24 \cdot sin\left(\frac{2\pi \cdot n}{T_s}\right) \tag{28}$$

The filter $H[z]$ is used to process a Gaussian Random Variable, where the filter is defined in Equation 30. The output of the filter will be observed in different window size defined as: $[N, 2N, 3N, 4N]$, with N defined as 25.

$$H[z] = \frac{1}{1 - \alpha \cdot z^{-1}} \tag{29}$$

Observing the results of the filter does not provide sufficient information about the implementation of the filter with a α varying. Figure 5 shows the estimate of the filter coefficient a. The reason for the estimate not resulting to sinusoidal waveform can be explained from Equation 31, Equation 32 and Equation 33.

$$y[n] = GRV(\mu, \sigma^2) \tag{30}$$

$$\begin{aligned} \mu &= E\big[x[n] + a \cdot y[n-1]\big] \\ &= E\big[a \cdot y[n-1]\big] \end{aligned} \tag{31}$$

Hence,

$$\sigma^2 = \big(y[n] - \mu\big)^2 \tag{32}$$

From Equation 2, a μ is dependent on the estimate of, and since a is **not** a constant, a remains as an estimate to be operated. From Equation 2, it can be seen that the value of a cannot be determined. Only the estimate of a can be determined.

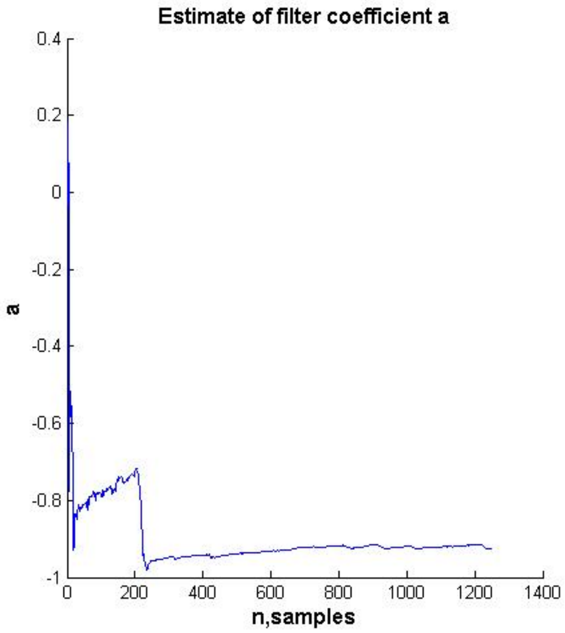

Figure 5: Estimated value of the filter coefficient a

Finally, the autocorrelation function illustrated in Figure 6-8 show that the output $y[n]$ is still Gaussian.

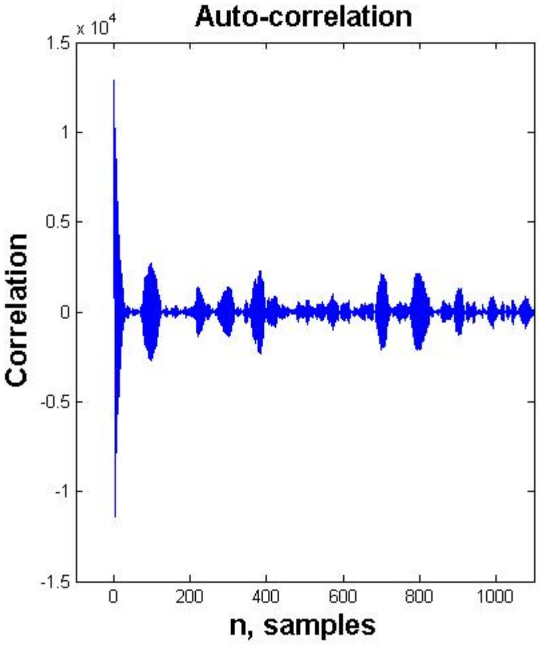

Figure 6: Auto-correlation function for the output of the filter with a window of 1 - 2500

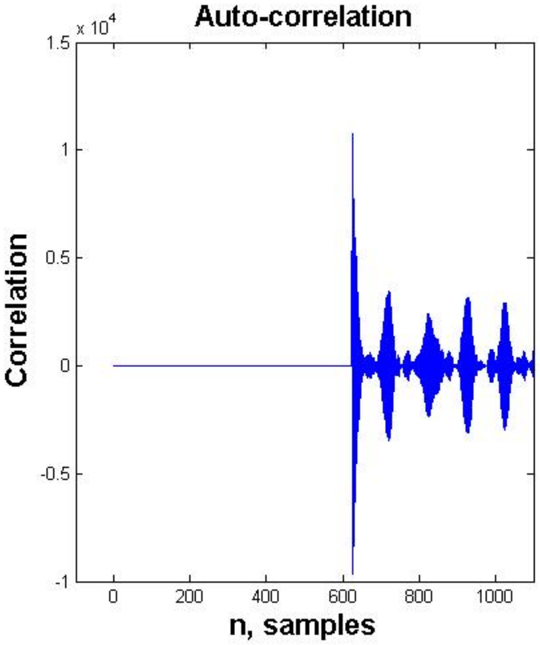

Figure 7: Auto-correlation function for the output of the filter with a window of 625 - 2500

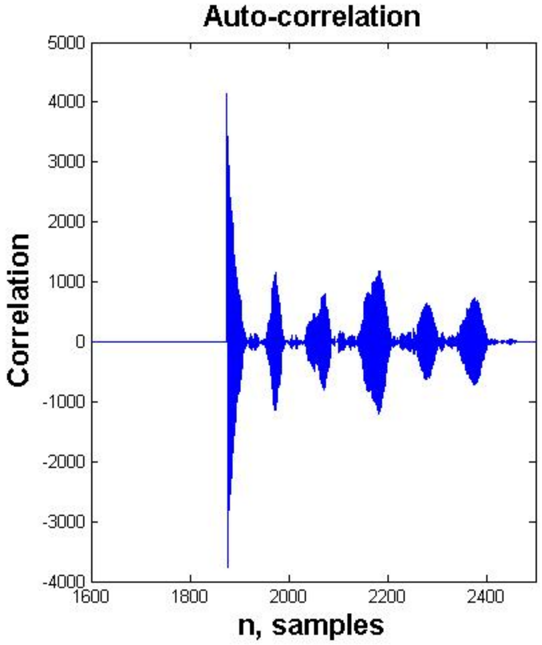

Figure 8: Auto-correlation function for the output of the filter with a window of 1875 - 2500

4.1 Observation

When attempting to estimate a, the output of the estimation converged to -0.75 which is the constant or the mean of the filter coefficient a. Therefore I conclude that the output remains a Gaussian Variable and to reinforce this statement, the auto-correlation function of the output $y[n]$ is an impulse, although it has small spikes along the way, it remains an impulse regardless of the window size.

5 Problem 5