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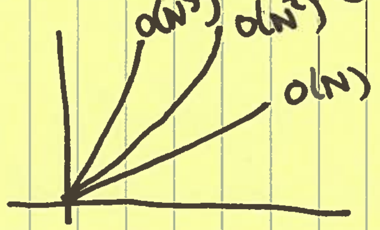
Lecture 24: Matrix form of DFT

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi kn/N} \qquad k = 0, N-1$$

computational complexity: $O(N^2)$



Linear Algebra:

$e^{-j2\pi nk/N}$ can be precomputed

$$x = \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{N-1} \end{bmatrix} \xrightarrow{DFT} X = \begin{bmatrix} X_0 \\ X_1 \\ \vdots \\ X_{N-1} \end{bmatrix}$$

$e^{-j2\pi kn/N}$

$$W_N = e^{-j2\pi/N}$$

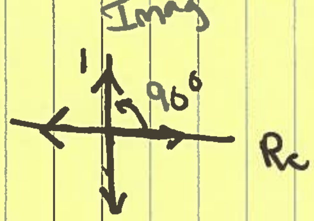
$$(W_N^n)^k = e^{-j2\pi/N\, nk}$$

$$(W_N^n) = e^{-j2\pi/N\, n}$$

$N = 4$

$$(W_4^n)^k = e^{-j\frac{2\pi}{4}nk}$$

$$e^{-j\frac{2\pi}{4}} = e^{-j\frac{\pi}{2}}$$



$n=1 \quad k=2$

$$e^{-j\frac{\pi}{2}(2)} = e^{-j\pi}$$

For $N=4$:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4^1 & W_4^2 & W_4^3 \\ 1 & W_4^2 & W_4^4 & W_4^6 \\ 1 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix}$$

$$X = \underline{A}\, x$$

For $N=4$

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

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The DFT is just a vector matrix multiplication. Amenable for parallel processing.

Modulo - N Reduction

partition the signal into N pt. blocks

$$x = [\,1, 2, -2, 3, 4, -2, -1, 1\,]^T$$

$$\tilde{x} = \begin{bmatrix} 1 \\ 2 \\ -2 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ -2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ -3 \\ 4 \end{bmatrix}$$

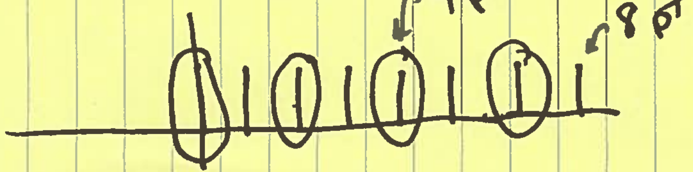
Do a 4-pt DFT on $\tilde{x}$

Same result if we did an 8-pt DFT and only retained 4 pts.

The reason this saves computation is because $O(8^2) >> O(4^2)$

4 pt.

8 pt



~~The DFT matrix is [illegible]~~

~~The modulo DFT matrix is $N \times L$ where L is the total length~~

The DFT matrix is $N \times L$ where L is the total length

The modulo DFT matrix is $N \times N$ when $N \ll L$

Alternate approach:

x y

Inverse DFT:

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi kn/N}$$

$$X = \underline{A}\, x$$

$$x = \underline{A}^{-1} X$$

$$x(n) = \frac{1}{N} \sum^{N-1} X(k)\, e^{j2\pi kn/N}$$

$$x \rightarrow \boxed{DFT} \rightarrow X \rightarrow \boxed{DFT} \rightarrow ?$$

$$x = \frac{1}{N}\left[DFT(X^*)\right]^*$$

taking the DFT twice gets you back to the original signal

The inverse DFT can also be computed using the matrix approach.

"Forward DFT" is often called "analysis"

"Inverse DFT" is often called "synthesis".

The DFT is exact, meaning:

$$x \rightarrow \boxed{DFT} \rightarrow X \rightarrow \boxed{IDFT} \rightarrow x$$