Lecture 10

$$y(n) = 0.5y(n-1) + x(n)$$

$$x(n) = \delta(n)$$

n	$x(n)$	$y(n-1)$	$y(n)$
0	1	0	1
1	0	1	.5
2	0	.5	.25
3	0	.25	.125
4	0	0	⋮

$$h(n) \Rightarrow H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n}$$

Z-transform of the transfer function

Z-transforms

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

Ex: $x(n) = \delta(n)$

$$X(z) = \sum_{n=-\infty}^{\infty} \delta(n) z^{-n}$$

$$= 1 \quad (n=0)$$

Properties:

Linearity:

$$a_1x_1(n) + a_2x_2(n) \Leftrightarrow a_1X_1(z) + a_2X_2(z)$$

Delay:

$$x(n) \Leftrightarrow X(z)$$

$$x(n-D) \Leftrightarrow z^{-D}X(z)$$

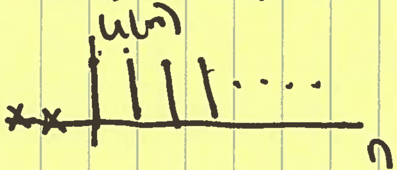
Convolution:

$$y(n) = h(n) * x(n)$$

$$Y(z) = H(z)X(z)$$

Example:

$\delta(n) = u(n) - u(n-1)$



$x(n) = u(n)$

$$X(z) = \sum_{n=-\infty}^{\infty} u(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} (1) z^{-n}$$

$\cancel{X(z)=}$

$x(n) = u(n-1)$

$$X(z) = z^{-1} \sum_{n=0}^{\infty} z^{-n}$$

$$Z\left[u(n) - u(n-1)\right] = \underbrace{\sum_{n=0}^{\infty} z^{-n}} - z^{-1} \sum_{n=0}^{\infty} z^{-n} = 1$$

$$X(z) - z^{-1} X(z) = 1$$

$$X(z) = \frac{1}{1 - z^{-1}} \Longleftrightarrow u(n)$$

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Region of Convergence

$$ROC \triangleq \sum_{n=-\infty}^{\infty} x(n) z^{-n} \neq \infty$$

$x(n) = [1, a, a^2, \ldots, a^n]$ $\quad x(n) = 0 \quad n < 0$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=0}^{\infty} a^n z^{-n}$$

$$\cancel{= \sum_{n=0}^{\infty} (a^n z^{-1})^n}$$

$$= \sum_{n=0}^{\infty} (a z^{-1})^n$$

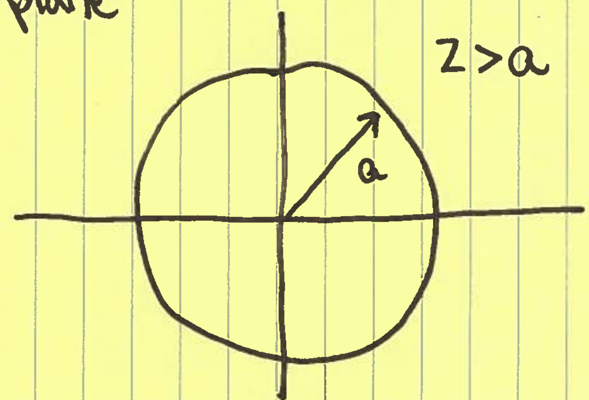
Use summation formula:

$$1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad x < 1$$

$$X(z) = Z[a^n u(n)] = \sum_{n=0}^{\infty} (a z^{-1})^n = \frac{1}{1 - a z^{-1}}$$

when $a z^{-1} < 1 \quad z^{-1} < 1/a \quad z > a$

z-plane



$z > a$

Ex:

$X(n) = -a^n u(-n-1)$

anti-causal version of $a^n u(n)$

$$X(z) = \frac{1}{1 - a z^{-1}}$$

But...

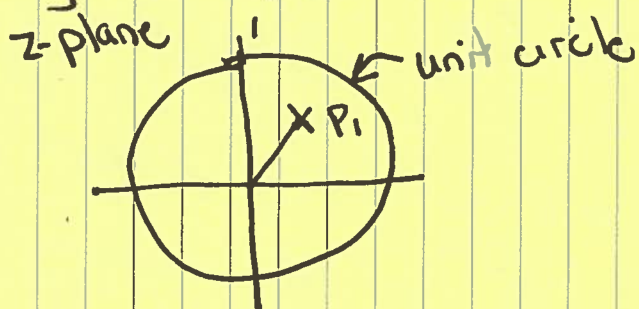
Stability

$$X(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots}$$

$$= \frac{A_1}{1 - p_1 z^{-1}} + \frac{A_2}{1 - p_2 z^{-1}} + \cdots$$

$$X(z) = \frac{A_1}{1 - p_1 z^{-1}} \Rightarrow x(n) = \left[(p_1)^n u(n)\right] A_1$$

only exists if $|p_1| < 1$

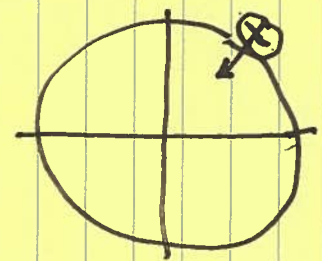


$|p_1| < 1 \Rightarrow$ stable

$$\sum_{n=0}^{\infty} |x(n)| < \infty$$

Note: the zeroes of a transfer function <u>do not</u> influence stability.

What do they influence? phase!



$x(n) = \delta(n)$ $\quad Z[x(n)] = X(z) = 1$

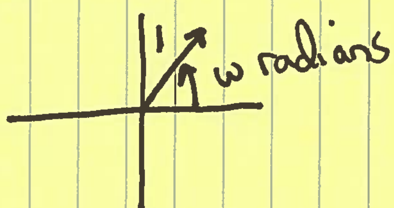
$x(n) = u(n)$ $\quad X(z) = \frac{1}{1 - z^{-1}}$

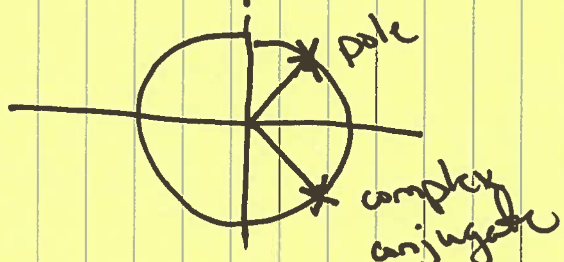
$x(n) = \cos \omega n \, u(n)$

$$X(z) = \sum_{n=0}^{\infty} \cos \omega(n) \, z^{-n}$$

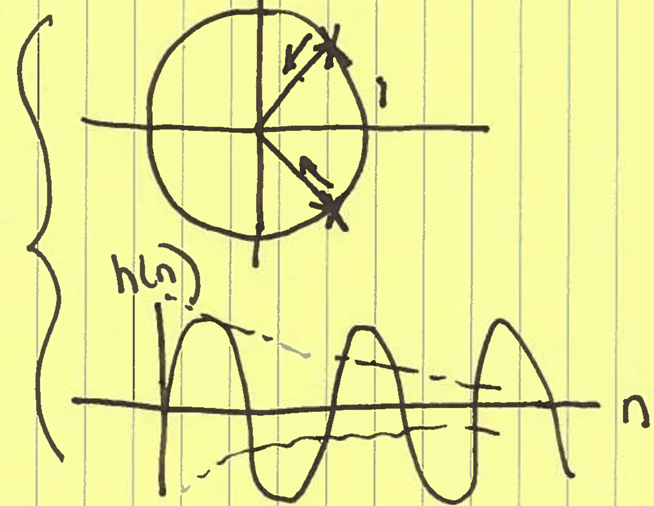
$$= \sum_{n=0}^{\infty} \tfrac{1}{2}\left[e^{j\omega n} + e^{-j\omega n}\right] z^{-n}$$

Note: $e^{j\omega n} = a^n$ where $a = e^{j\omega}$





z plane



$e^{-an} \cos \omega n$