Basic Problems

23. (a) Proof:

$$E[\hat{\sigma}^2] = E\left[\frac{1}{N}\sum_{k=1}^{N}(x_k - \hat{\mu})^2\right] = \frac{1}{N}\sum_{k=1}^{N} E[(x_k - \hat{\mu})^2]$$
$$= \frac{1}{N}\sum_{k=1}^{N}\left(E[x_k^2] - 2E[x_k\hat{\mu}] + E[\hat{\mu}^2]\right)$$
$$= \frac{1}{N}\sum_{k=1}^{N}\left(\sigma^2 + \mu^2 - 2\times\frac{1}{N}(N\mu^2 + \sigma^2) + \frac{1}{N^2}(N^2\mu^2 + N\sigma^2)\right)$$
$$= \frac{1}{N}\sum_{k=1}^{N}\left(\frac{N-1}{N}\sigma^2\right) = \frac{N-1}{N}\sigma^2$$

(b) Proof:

$$E[\hat{\sigma}^2] = \frac{1}{N}\sum_{k=1}^{N} E[(x_k - \mu)^2] = \frac{1}{N}\sum_{k=1}^{N}\left(E[x_k^2] - 2\mu E[x_k] + \mu^2\right)$$
$$= \frac{1}{N}\sum_{k=1}^{N}\left(\mu^2 + \sigma^2 - 2\mu^2 + \mu^2\right)$$
$$= \sigma^2$$

24. (a) Comments:
See script output for details.

(b) See plot below.

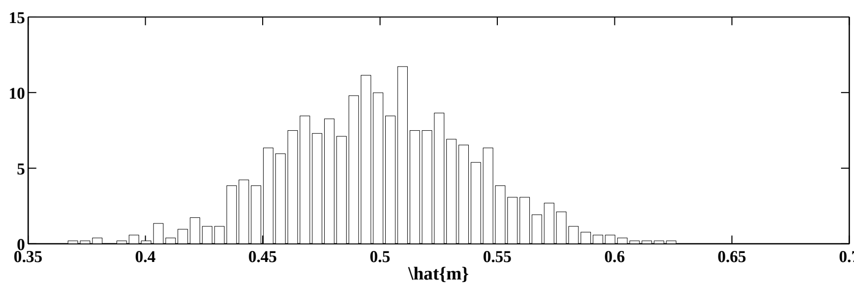

FIGURE 14.34: Plot of empirical pdf of the sample mean.

MATLAB script:

33. Solution:

The variance of the estimator is

$$J = E[(X - \hat{X})^2] = E[(X - aY - b)^2] = E[X^2] + E[(aY + b)^2] - 2E[X(aY + b)]$$
$$= (\mu_x^2 + \sigma_x^2) + a^2(\mu_y^2 + \sigma_y^2) + 2ab\mu_y + b^2 - 2a(\sigma_{xy} + \mu_x\mu_y) - 2b\mu_x$$

Take the partial derivatives with respect to a and b and constrain the results equal to zero, we have

$$\frac{\partial J}{\partial a} = 2a(\mu_y^2 + \sigma_y^2) + 2b\mu_y - 2\sigma_{xy} - 2\mu_x\mu_y = 0 \quad \text{(P33a)}$$

$$\frac{\partial J}{\partial b} = 2a\mu_y + 2b - 2\mu_x = 0 \quad \text{(P33b)}$$

Solve Eq. (P33b), we have

$$b = \mu_x - a\mu_y$$

Substitute b back to Eq.(P33a) and solve for a, we have

$$a = \frac{\sigma_{xy}}{\sigma_y^2}$$