```
hfa = figconfg('P0702a','long');
subplot(121)
stem(k,abs(ck),'filled');hold on
stem(k,abs(ck_approx),'filled','color','red');
xlabel('k','fontsize',LFS)
title('Magnitude Response','fontsize',TFS)
legend('c_k','\tilde{c}_k','location','best')
subplot(122)
stem(k,angle(ck),'filled');hold on
stem(k,angle(ck_approx),'filled','color','red');
xlabel('k','fontsize',LFS)
title('Phase Response','fontsize',TFS)
legend('c_k','\tilde{c}_k','location','best')
```

3. (a) Solution:
   The DTFT of $(0.9)^n u[n]$ is:

$$\frac{1}{1-0.9\mathrm{e}^{-\mathrm{j}\omega}}$$

   The DTFT of $x[n]$ is:

$$\begin{aligned}\tilde{X}(\mathrm{e}^{\mathrm{j}\omega}) &= (-\mathrm{j})\frac{\mathrm{d}}{\mathrm{d}\omega}\left(\frac{1}{1-0.9\mathrm{e}^{-\mathrm{j}\omega}}\right)\\ &= \frac{0.9\mathrm{e}^{-\mathrm{j}\omega}}{(1-0.9\mathrm{e}^{-\mathrm{j}\omega})^2}\end{aligned}$$

   (b) See plot below.
   (c) See plot below.
   (d) See plot below.

MATLAB script:

```
% P0703: Numerical DFT approximation of DTFT
close all; clc
w = linspace(0,2,1000)*pi;
X = 0.9*exp(-j*w)./(1-0.9*exp(-j*w)).^2;
N = 20; % Part (b)
% N = 50; % Part (c)
% N = 100; % Part (d)
n = 0:N-1;
```

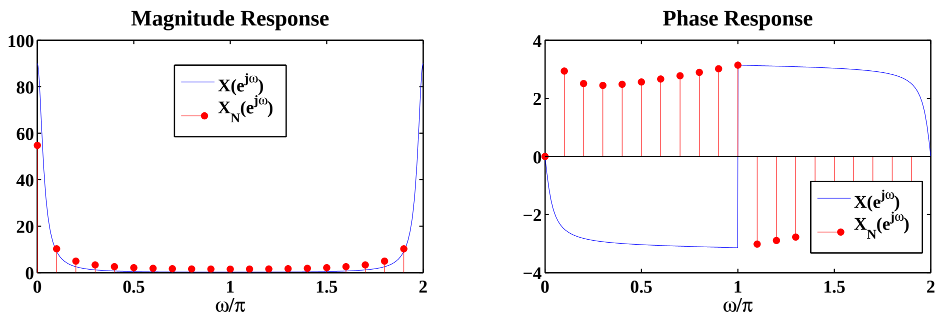


FIGURE 7.6: Magnitude and phase responses of $\tilde{X}(e^{j\omega})$ and $\tilde{X}_N(e^{j\omega})$ when $N = 20$.

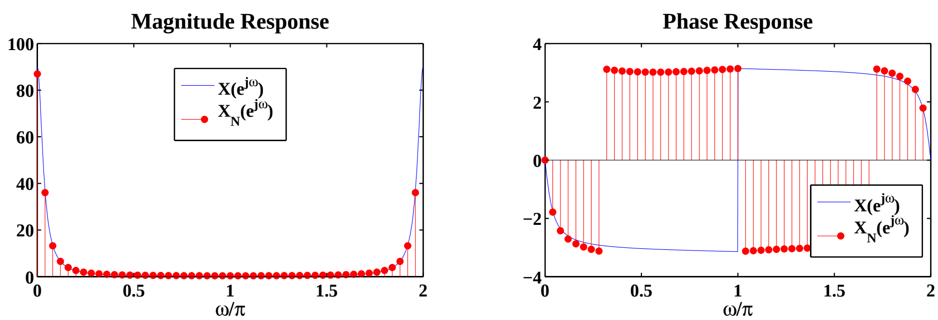


FIGURE 7.7: Magnitude and phase responses of $\tilde{X}(e^{j\omega})$ and $\tilde{X}_N(e^{j\omega})$ when $N = 50$.

```
xn = n.*0.9.^n;
XN = fft(xn);
wk = 2/N*(0:N-1);
%% Plot:
hfa = figconfg('P0703a','long');
subplot(121)
plot(w/pi,abs(X));hold on
stem(wk,abs(XN),'filled','color','red');
xlabel('\omega/\pi','fontsize',LFS)
title('Magnitude Response','fontsize',TFS)
legend('X(e^{j\omega})','X_N(e^{j\omega})','location','best')
subplot(122)
plot(w/pi,angle(X));hold on
```

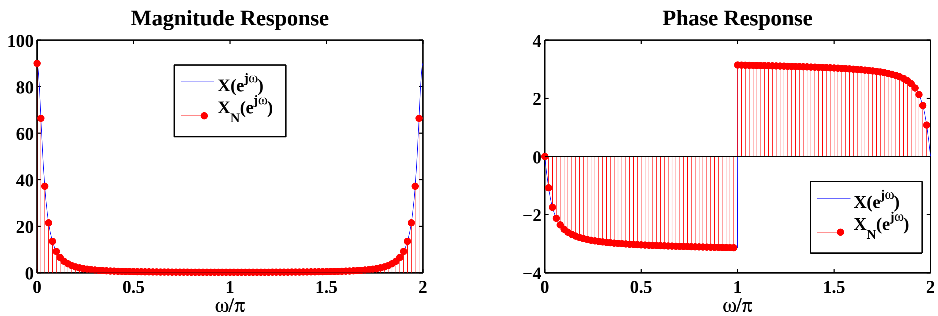


FIGURE 7.8: Magnitude and phase responses of $\tilde{X}(\mathrm{e}^{\mathrm{j}\omega})$ and $\tilde{X}_N(\mathrm{e}^{\mathrm{j}\omega})$ when $N = 100$.

```
stem(wk,angle(XN),'filled','color','red');
xlabel('\omega/\pi','fontsize',LFS)
title('Phase Response','fontsize',TFS)
legend('X(e^{j\omega})','X_N(e^{j\omega})','location','best')
```

4. (a) Solution:
The $N \times N$ DFT matrix is:

$$\boldsymbol{W}_N = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & W_N & \cdots & W_N^{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{N-1} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix}$$

The kth column of $\boldsymbol{W}_N$ is $\boldsymbol{w}_k = [1 W_N^k \dots W_N^{N-1} k]^T$.
The i,jth element of $\boldsymbol{W}_N^2$ is:

$$\begin{aligned} \left(\boldsymbol{W}_N^2\right)_{i,j} &= \left(\boldsymbol{W}_N^T \boldsymbol{W}_N\right)_{i,j} = \boldsymbol{w}_i^T \boldsymbol{w}_j \\ &= \begin{cases} 0, & i+j \neq N \\ N, & i+j = N \end{cases} \end{aligned}$$

Hence, we proved that

$$\boldsymbol{W}_N^2 = \begin{bmatrix} 0 & \cdots & 0 & N \\ \vdots & \iddots & N & 0 \\ 0 & \iddots & \iddots & \vdots \\ N & 0 & \cdots & 0 \end{bmatrix} = N \boldsymbol{J}_N$$

8. (a) See plot below.
   (b) See plot below.
   (c) See plot below.

   MATLAB script:

```
% P0708: Regenerate Figure~7.5 and Example 7.3
close all; clc
N = 16; a = 0.9; % Part (a)
% N = 8; a = 0.8; % Part (b)
% N = 64; a = 0.8; % Part (c)
wk = 2*pi/N*(0:N-1);
Xk = 1./(1-a*exp(-j*wk));
xn = real(ifft(Xk));
w = linspace(0,2,1000)*pi;
X = fft(xn,length(w));
X_ref = 1./(1-a*exp(-j*w));
n = 0:N-1;
xn_ref = a.^n;
%% Plot:
hfa = figconfg('P0708a','small');
plot(w/pi,abs(X_ref),'color','black'); hold on
plot(w/pi,abs(X))
stem(wk/pi,abs(Xk),'filled');
ylim([0 max(abs(X))])
xlabel('\omega/\pi','fontsize',LFS)
ylabel('Magnitude','fontsize',LFS)
title('Magnitude of Spectra','fontsize',TFS)
hfb = figconfg('P0708b','small');
plot(n,xn,'.'); hold on
plot(n,xn_ref,'.','color','black')
ylim([0 1.1*max(xn)])
xlim([0 N-1])
xlabel('Time index (n)','fontsize',LFS)
ylabel('Amplitude','fontsize',LFS)
title('Signal Amplitudes','fontsize',TFS)
legend('x[n]','\tilde{x}[n]','location','northeast')
```

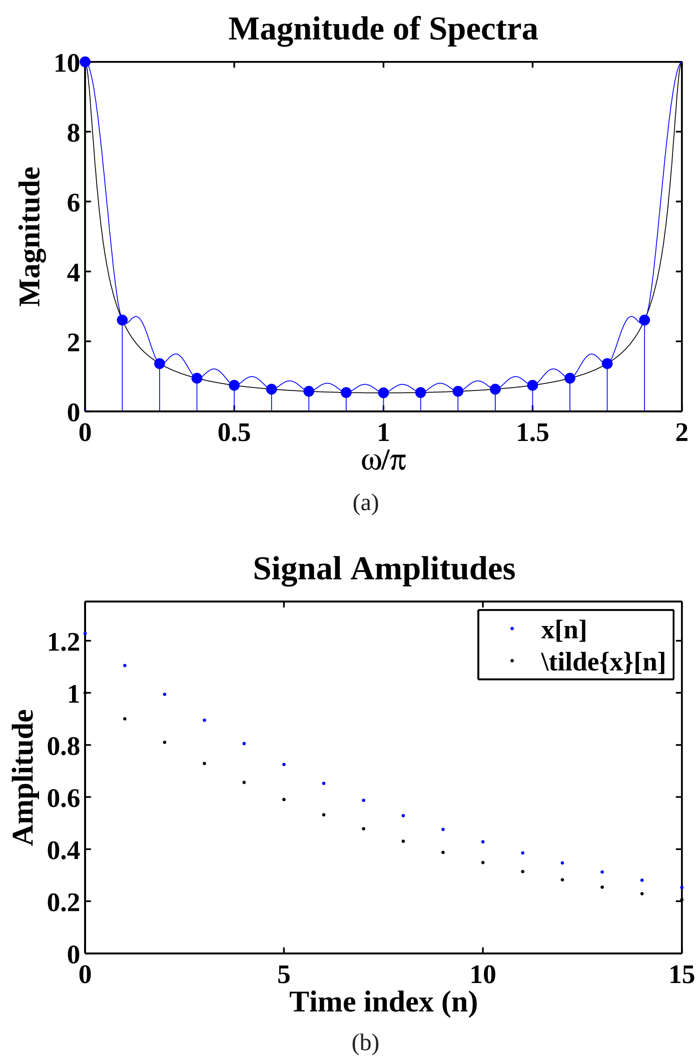
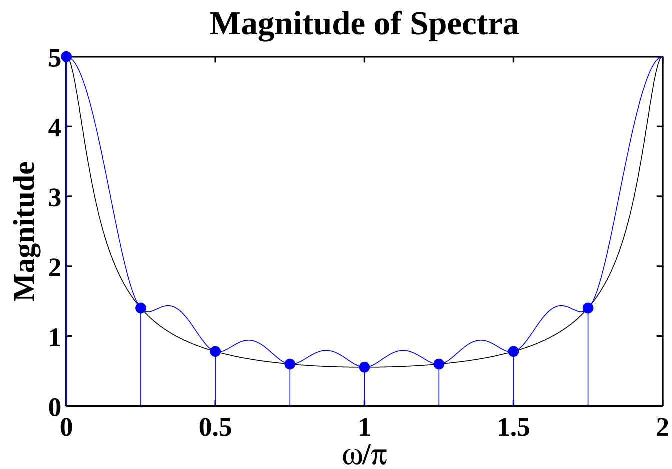
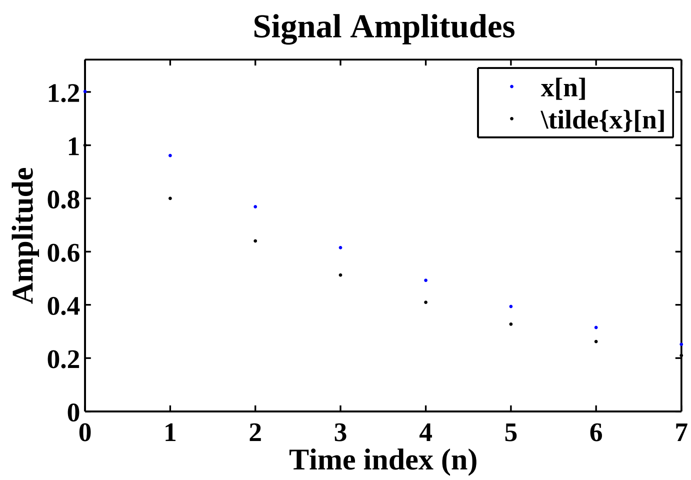


FIGURE 7.9: (a) Magnitude response of the DTFT signal. (b) Time sequence and reconstructed time sequence.



(a)



(b)

FIGURE 7.10: (a) Magnitude response of the DTFT signal and (b) time sequence and reconstructed time sequence for $a = 0.8$ and $N = 8$.

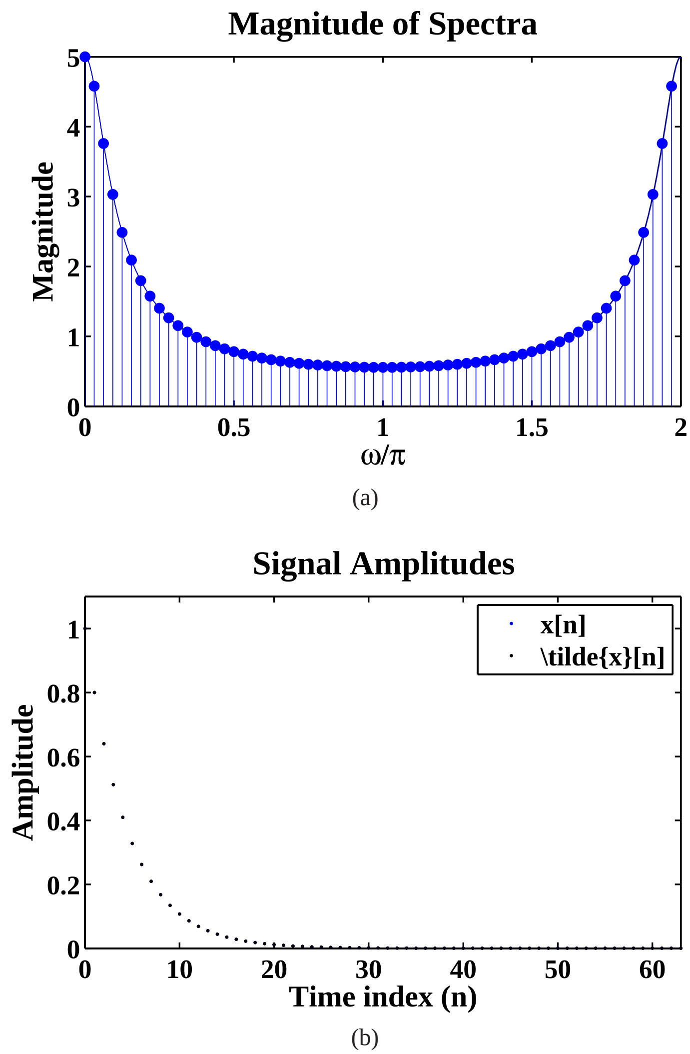


FIGURE 7.11: (a) Magnitude response of the DTFT signal and (b) time sequence and reconstructed time sequence for $a = 0.8$ and $N = 64$.

13. (a) Proof:
If k is even and N is even, the correspondent DFT is:

$$\begin{aligned}
X[k] &= \sum_{n=0}^{N-1} x[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}nk} = \sum_{n=0}^{N/2-1} x[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}nk} + \sum_{n=N/2}^{N-1} x[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}nk} \\
&= \sum_{n=0}^{N/2-1} \left(x[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}nk} + x[n+\frac{N}{2}]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}(n+\frac{N}{2})k} \right) \\
&= \sum_{n=0}^{N/2-1} \left(x[n] - x[n] \right) \mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}nk} = 0
\end{aligned}$$

(b) Proof:
If $N = 4m$, $k = 4\ell$, the correspondent DFT is:

$$\begin{aligned}
X[4\ell] &= \sum_{n=0}^{N-1} x[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}n(4\ell)} = \sum_{n=0}^{\frac{N}{4}-1} x[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}n(4\ell)} \\
&+ \sum_{n=\frac{N}{4}}^{\frac{2N}{4}-1} x[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}n(4\ell)} + \sum_{n=\frac{2N}{4}}^{\frac{3N}{4}-1} x[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}n(4\ell)} + \sum_{n=\frac{3N}{4}}^{N-1} x[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}n(4\ell)} \\
&= \left(\sum_{n=0}^{\frac{N}{4}-1} x[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}n(4\ell)} + \sum_{n=0}^{\frac{N}{4}-1} x[n+\frac{N}{4}]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}(n+\frac{N}{4})(4\ell)} \right) \\
&+ \left(\sum_{n=\frac{2N}{4}}^{\frac{3N}{4}-1} x[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}n(4\ell)} + \sum_{n=\frac{2N}{4}}^{\frac{3N}{4}-1} x[n+\frac{N}{4}]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}(n+\frac{N}{4})(4\ell)} \right) \\
&= \sum_{n=0}^{\frac{N}{4}-1} \left(x[n] + x[n+\frac{N}{4}] \right) \mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}n(4\ell)} + \sum_{n=\frac{2N}{4}}^{\frac{3N}{4}-1} \left(x[n] + x[n+\frac{N}{4}] \right) \mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}n(4\ell)} \\
&= 0
\end{aligned}$$

14. (a) Solution:
Solving the circular convolution using hand calculation:

$$\begin{bmatrix} 2 & 0 & -1 & 1 & -1 \\ -1 & 2 & 0 & -1 & 1 \\ 1 & -1 & 2 & 0 & -1 \\ -1 & 1 & -1 & 2 & 0 \\ 0 & -1 & 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ 0 \\ 6 \\ 7 \end{bmatrix}$$

(b) See script below.

(c) See script below.

MATLAB script:

```
% P0714: Circular convolution
close all; clc
xn1 = 1:5;
xn2 = [2 -1 1 -1];
%% Part (b):
xn = circonv(xn1',[xn2 0]');
%% Part (c):
N = max(length(xn1),length(xn2));
Xk1 = fft(xn1,N);
Xk2 = fft(xn2,N);
Xk = Xk1.*Xk2;
xn_dft = ifft(Xk);
```

15. (a) Proof:
$X_4[K]$ can be obtained by frequency sampling of $X_3[k]$, hence in the time domain, according the aliasing equation, we have

$$x_4[n] = \sum_{\ell=-\infty}^{\infty} x_3[n + \ell N]$$

(b) Proof:
When $N \geq L$, there is no time aliasing, we conclude

$$x_4[n] = x_3[n], \quad \text{for} \quad 0 \leq n \leq L$$

When $\max(N_1, N_2) \leq N < L$, since $L = N_1 + N_2 - 1 \leq 2N - 1$, we conclude that

$$x_4[n] = x_3[n] + x_3[n + N], \quad \text{for} \quad 0 \leq n \leq N - 1$$

Hence, we proved the equation (**??**).

(c) MATLAB script:

```
% P0715: Verify formula in Problem 0715
close all; clc
xn1 = 1:4;
xn2 = 4:-1:1;
```

Linearity.

$$\begin{aligned} w_c(t)\cdot(a_1x_{c1}(t)+a_2x_{c2}(t)) &= a_1w_c(t)x_{c1}(t)+a_2w_c(t)x_{c2}(t)\\ &= a_1\tilde{x}_{c1}(t)+a_2\tilde{x}_{c2}(t)\end{aligned}$$

Time-varying.

$$\text{In general,}\qquad w_c(t-\tau)x_c(t-\tau)\neq w_c(t)x_c(t)$$

Hence, $\tilde{x}_c(t-\tau)\neq\tilde{x}_c(t)$.

(b) Proof:

$$\tilde{x}[n]=w[n]x[n] \tag{7.160}$$

If $0\le t\le T_0$, and $0\le n\le L$, we have

$$\begin{aligned}&\tilde{x}_c(nT)=w_c(nT)x_c(nT)=x_c(nT)=x[n]\\ &\tilde{x}[n]=w[n]x[n]=x[n]\end{aligned}$$

If $t>T_0$, and $N>L$, we have

$$\tilde{x}_c(nT)=\tilde{x}[n]=0$$

21. Proof:

$$\hat{X}_c(\mathrm{j}\Omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X_c(\mathrm{j}\theta)W_c(j(\Omega-\theta))\mathrm{d}\theta \tag{7.170}$$

The CTFT of $\hat{x}_c(t)$ is:

$$\begin{aligned}\hat{X}_c(\mathrm{j}\Omega) &= \int_{-\infty}^{\infty}\hat{x}_c(t)\mathrm{e}^{-\mathrm{j}\Omega t}\mathrm{d}t=\int_{-\infty}^{\infty}w_c(t)x_c(t)\mathrm{e}^{-\mathrm{j}\Omega t}\mathrm{d}t\\ &= \int_{-\infty}^{\infty}w_c(t)\mathrm{e}^{-\mathrm{j}\Omega t}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}X_c(\mathrm{j}\theta)\mathrm{e}^{\mathrm{j}\theta t}\mathrm{d}\theta\right)\mathrm{d}t\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}X_c(\mathrm{j}\theta)\left(\int_{-\infty}^{\infty}w_c(t)\mathrm{e}^{-\mathrm{j}(\Omega-\theta)t}\right)\mathrm{d}\theta\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}X_c(\mathrm{j}\theta)W_c(j(\Omega-\theta))\mathrm{d}\theta\end{aligned}$$

22. Proof:

$$\text{Scaling Property:}\quad x_c(at)\overset{\text{CTFT}}{\longleftrightarrow}\frac{1}{|a|}X_c\left(\frac{\mathrm{j}\Omega}{a}\right) \tag{7.172}$$

The CTFT of $x_c(at)$ is:

$$\int_{-\infty}^{\infty}x_c(at)\mathrm{e}^{-\mathrm{j}\Omega t}\mathrm{d}t=\frac{1}{a}\int_{-\infty}^{\infty}x_c(at)\mathrm{e}^{-\mathrm{j}\frac{\Omega}{a}at}\mathrm{d}at$$

If $a > 0$, we have

$$\frac{1}{a}\int_{-\infty}^{\infty} x_c(at)\mathrm{e}^{-\mathrm{j}\frac{\Omega}{a}at}\mathrm{d}at = \frac{1}{a}\int_{-\infty}^{\infty} x_c(t)\mathrm{e}^{-\mathrm{j}\frac{\Omega}{a}t}\mathrm{d}t = \frac{1}{a}X_c\left(\frac{\mathrm{j}\Omega}{a}\right)$$

If $a < 0$, we have

$$\frac{1}{a}\int_{\infty}^{-\infty} x_c(at)\mathrm{e}^{-\mathrm{j}\frac{\Omega}{a}at}\mathrm{d}at = \frac{1}{a}\int_{\infty}^{-\infty} x_c(t)\mathrm{e}^{-\mathrm{j}\frac{\Omega}{a}t}\mathrm{d}t = -\frac{1}{a}X_c\left(\frac{\mathrm{j}\Omega}{a}\right)$$

Hence, we proved the scaling property.

23. Proof:

$$\left|\int_{-\infty}^{\infty} x_{c1}(t)x_{c2}(t)\mathrm{d}t\right|^2 \le \int_{-\infty}^{\infty} |x_{c1}(t)|^2\mathrm{d}t \int_{-\infty}^{\infty} |x_{c2}(t)|^2\mathrm{d}t \qquad (7.179)$$

Suppose a is a real number, define function $p(a)$ as

$$p(a) = \int_{-\infty}^{\infty} \left(a \cdot x_{c1}(t) + x_{c2}(t)\right)^2 \mathrm{d}t = Aa^2 + 2Ba + C \ge 0$$

where

$$A = \int_{-\infty}^{\infty} x_{c1}^2(t)\mathrm{d}t, \quad B = \int_{-\infty}^{\infty} x_{c1}(t)x_{c2}(t)\mathrm{d}t, \quad C = \int_{-\infty}^{\infty} x_{c2}^2(t)\mathrm{d}t.$$

Since we have $4B^2 - 4AC \le 0$, that is $B^2 \le AC$,

$$\left(\int_{-\infty}^{\infty} x_{c1}(t)x_{c2}(t)\mathrm{d}t\right)^2 \le \int_{-\infty}^{\infty} x_{c1}^2(t)\mathrm{d}t \int_{-\infty}^{\infty} x_{c2}^2(t)\mathrm{d}t$$

24. Proof:
The CTFT of generic window is:

$$W(\mathrm{j}\Omega) = aW_{\mathrm{R}}(\mathrm{j}\Omega) + bW_{\mathrm{R}}(\mathrm{j}(\Omega - 2\pi)/T_0) + bW_{\mathrm{R}}(\mathrm{j}(\Omega + 2\pi)/T_0) \qquad (7.189)$$

The ICTFT is:

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\infty}^{\infty} \left[aW_{\mathrm{R}}(\mathrm{j}\Omega) + bW_{\mathrm{R}}(\mathrm{j}(\Omega - 2\pi)/T_0) + bW_{\mathrm{R}}(\mathrm{j}(\Omega + 2\pi)/T_0)\right]\mathrm{e}^{\mathrm{j}\Omega t}\mathrm{d}\Omega \\
&\quad = aw_R(t) + b\mathrm{e}^{\mathrm{j}\frac{2\pi}{T_0}t}w_R(t) + b\mathrm{e}^{-\mathrm{j}\frac{2\pi}{T_0}t}w_R(t) \\
&\quad = \left[a + 2b\cos\left(\frac{2\pi}{T_0}t\right)\right]w_R(t)
\end{aligned}$$

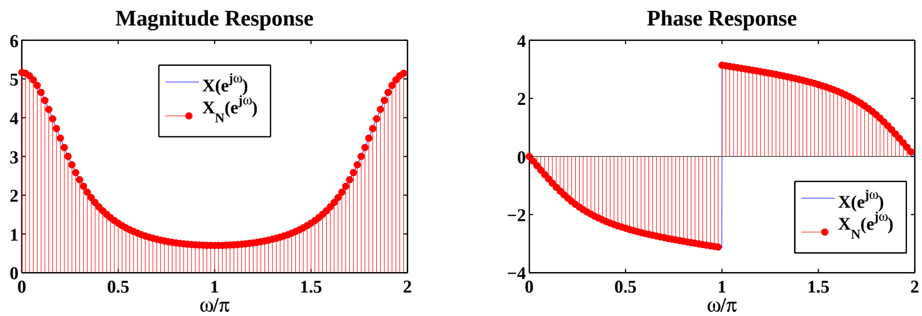


FIGURE 7.21: Magnitude and phase responses of $\tilde{X}(e^{j\omega})$ and $\tilde{X}_N(e^{j\omega})$ when $N = 100$.

```
title('Magnitude Response','fontsize',TFS)
legend('X(e^{j\omega})','X_N(e^{j\omega})','location','best')
subplot(122)
plot(w/pi,angle(X));hold on
stem(wk,angle(XN),'filled','color','red');
xlabel('\omega/\pi','fontsize',LFS)
title('Phase Response','fontsize',TFS)
legend('X(e^{j\omega})','X_N(e^{j\omega})','location','best')
```

30. tba

31. (a) See plot below.
    (b) See plot below.
    (c) See plot below.
    (d) See plot below.

    MATLAB script:

```
% P0731: Compute and plot DFT and IDFT
close all; clc
%% Part (a):
N = 8;
n = 0:N-1;
xn = zeros(size(n));
xn(1) = 1;

%% Part (b):
```

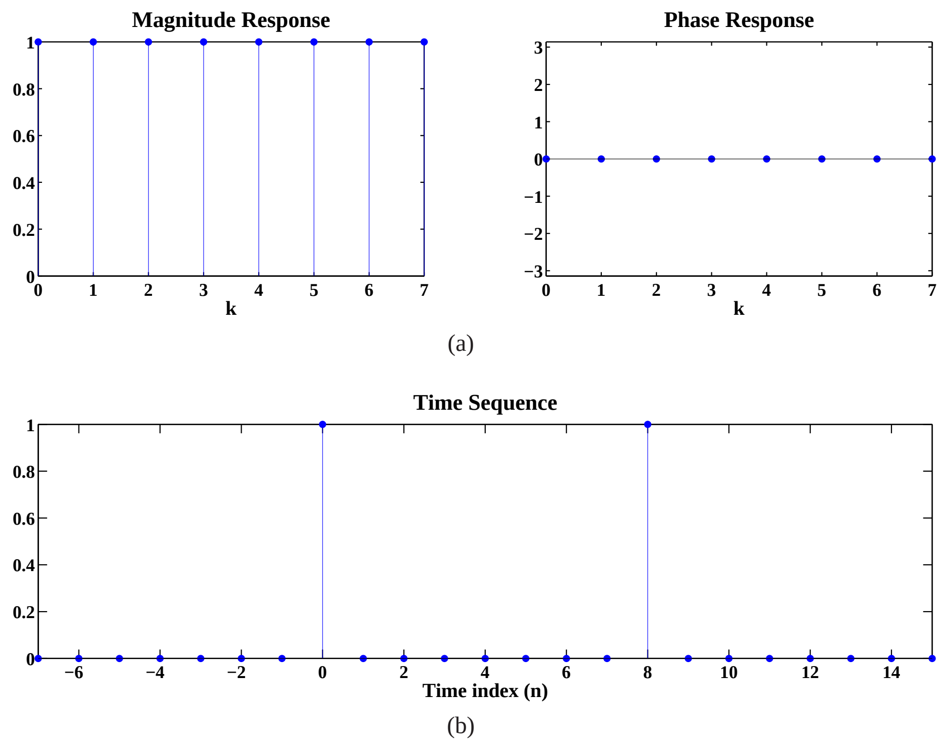
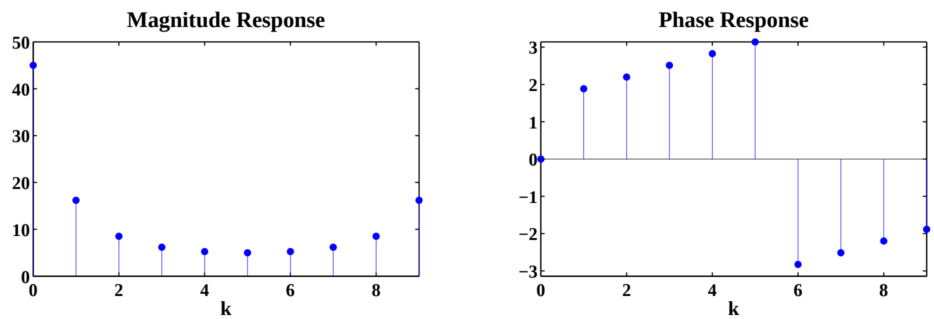
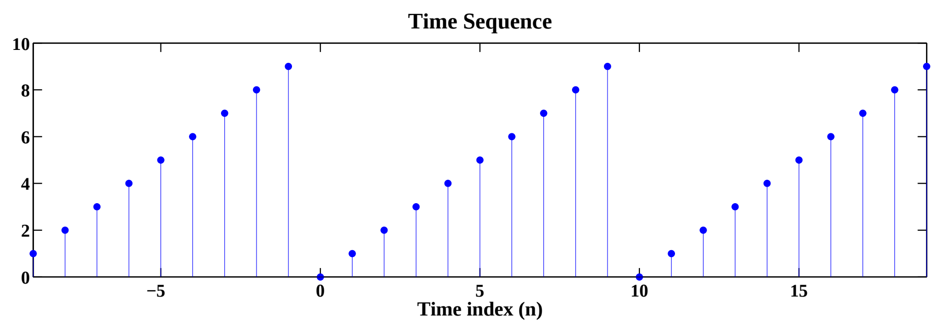


FIGURE 7.22: N-point (a) DFT and (b) IDFT of $x[n] = \delta[n]$, $N = 8$ in the range $-(N-1) \leq n \leq (2N-1)$.

```
% N = 10;
% n = 0:N-1;
% xn = n;

%% Part (c):
% N = 30;
% n = 0:N-1;
% xn = cos(6*pi*n/15);

%% Part (d):
% N = 30;
% n = 0:N-1;
% xn = cos(0.1*pi*n);

Xk = fft(xn);
```



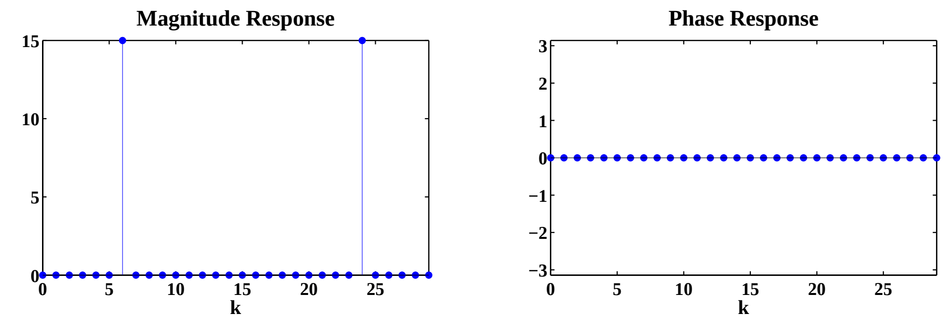
(a)



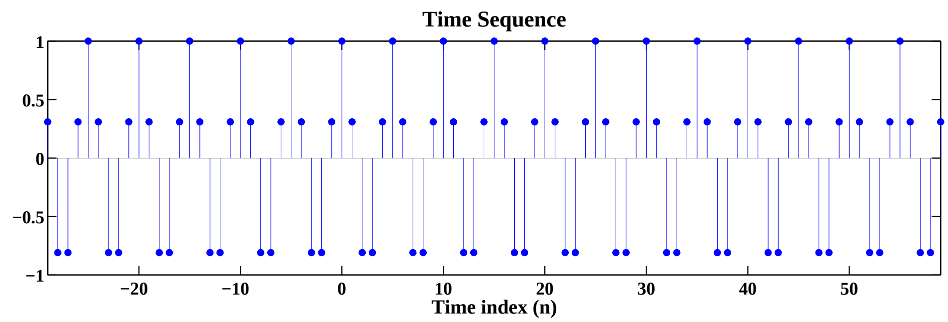
(b)

FIGURE 7.23: N-point (a) DFT and (b) IDFT of $x[n] = n$, $N = 10$ in the range $-(N-1) \leq n \leq (2N-1)$.

```
ind = abs(Xk) < 1e-10;
Xk(ind) = 0;
xn_ref = ifft(Xk);
nn = -(N-1):2*N-1;
xn_plot = xn_ref(mod(nn,N)+1);

%% Plot:
hfa = figconfg('P0731a','long');
subplot(121)
stem(n,abs(Xk),'filled');
xlim([0 N-1])
xlabel('k','fontsize',LFS)
title('Magnitude Response','fontsize',TFS)
subplot(122)
stem(n,angle(Xk),'filled');
```



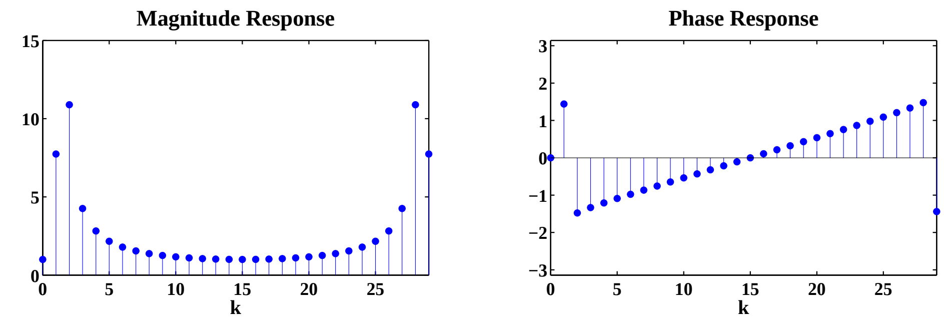
(a)



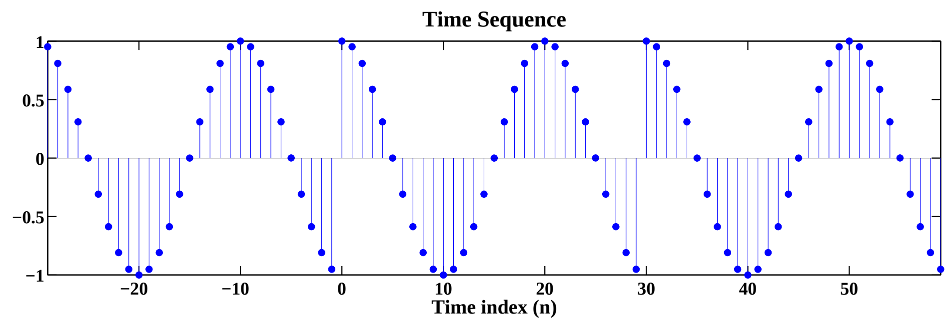
(b)

FIGURE 7.24: N-point (a) DFT and (b) IDFT of $x[n] = \cos(6\pi n/15)$, $N = 30$ in the range $-(N-1) \leq n \leq (2N-1)$.

```
xlim([0 N-1])
ylim([-pi pi])
xlabel('k','fontsize',LFS)
title('Phase Response','fontsize',TFS)

hfb = figconfg('P0731b','long');
stem(nn,xn_plot,'filled')
xlim([nn(1) nn(end)])
xlabel('Time index (n)','fontsize',LFS)
title('Time Sequence','fontsize',TFS)
```



(a)



(b)

FIGURE 7.25: N-point (a) DFT and (b) IDFT of $x[n] = \cos(0.1\pi n)$, $N = 30$ in the range $-(N-1) \leq n \leq (2N-1)$.

32. (a) Solution: The DFS of $\tilde{x}[n]$ and $\tilde{x}_3[n]$ can be written as:

$$\tilde{X}[k] = X[\langle k\rangle_N] = \sum_{n=0}^{N-1} \tilde{x}[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}n\langle k\rangle_N}$$

$$\tilde{X}_3[k] = X_3[\langle k\rangle_{3N}] = \sum_{n=0}^{3N-1} \tilde{x}[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{3N}n\langle k\rangle_{3N}}$$

(c) By applying the correlation property, the DFT of $x_3[n]$ is:

$$X_3[k] = X[k]X^*[k] = |X[k]|^2$$

(d) By applying the windowing property, the DFT of $x_4[n]$ is:

$$X_4[k] = \frac{1}{9} X[k] \, 9 \, X[k]$$

(e) By applying the frequency-shifting property, the DFT of $x_5[n]$ is:

$$X_5[k] = X[\langle k+2 \rangle_9]$$

39. (a) Proof:

$$X[0] = \sum_{n=0}^{N-1} x[n] W_N^{n \cdot 0} = \sum_{n=0}^{N-1} x[n]$$

which is real-valued since $x[n]$ is real-valued.

(b) Proof:
If $k = 0$, since $x[0]$ is real, we have

$$X[0] = X^*[0]$$

If $1 \leq k \leq N-1$, we have

$$\begin{aligned} X[\langle N-k \rangle_N] &= \sum_{n=0}^{N-1} x[n] W_N^{n\langle N-k \rangle_N} = \sum_{n=0}^{N-1} x[n] W_N^{n(N-k)} \\ &= \sum_{n=0}^{N-1} x[n] W_N^{-nk} = \left(\sum_{n=0}^{N-1} x[n] W_N^{nk} \right)^* \\ &= X^*[k] \end{aligned}$$

Hence, we proved $X[\langle N-k \rangle_N] = X^*[k]$ for every k.

(c) Proof:

$$X[N/2] = \sum_{n=0}^{N-1} x[n] W_N^{n\frac{N}{2}} = \sum_{n=0}^{N-1} x[n] \mathrm{e}^{-\mathrm{j} n\pi} = \sum_{n=0}^{N-1} x[n] \cos(n\pi)$$

which is real-valued since $x[n]$ and $\cos(n\pi)$ are both real-valued.