CHAPTER 6

Sampling of Continuous-Time Signals

Tutorial Problems

1. (a) Solution:

$$x_c(t) = \frac{5}{2}e^{j\frac{\pi}{6}}e^{j200\pi t} + \frac{5}{2}e^{-j\frac{\pi}{6}}e^{-j200\pi t} + \frac{2}{j}e^{j300\pi t} - \frac{2}{j}e^{-j300\pi t}$$

The spectra of $x_c(t)$ is:

$$Xc(j\Omega) = \begin{cases} \frac{5}{2}e^{j\frac{\pi}{6}}, & \Omega = 200\pi \\ \frac{5}{2}e^{-j\frac{\pi}{6}}, & \Omega = -200\pi \\ \frac{2}{j}, & \Omega = 300\pi \\ -\frac{2}{j}, & \Omega = -300\pi \\ 0, & \text{elsewhere} \end{cases}$$

The spectra $X(e^{j\omega})$ of $x[n]$ is:

$$X(e^{j\omega})\big|_{\omega=\Omega T} = F_s \sum_{k=-\infty}^{\infty} X_c(j\Omega - j2\pi k F_s)$$

$$X(e^{j\omega})\big|_{\omega=2\pi FT} = F_s \sum_{k=-\infty}^{\infty} X_c[j2\pi(F - kF_s)]$$

The signal can recovered for $x[n]$ if $F_s = 1$ KHz.

(b) Solution:
The signal can recovered for $x[n]$ if $F_s = 500$ Hz.

(c) Solution:
The signal can NOT recovered for $x[n]$ if $F_s = 100$ Hz.

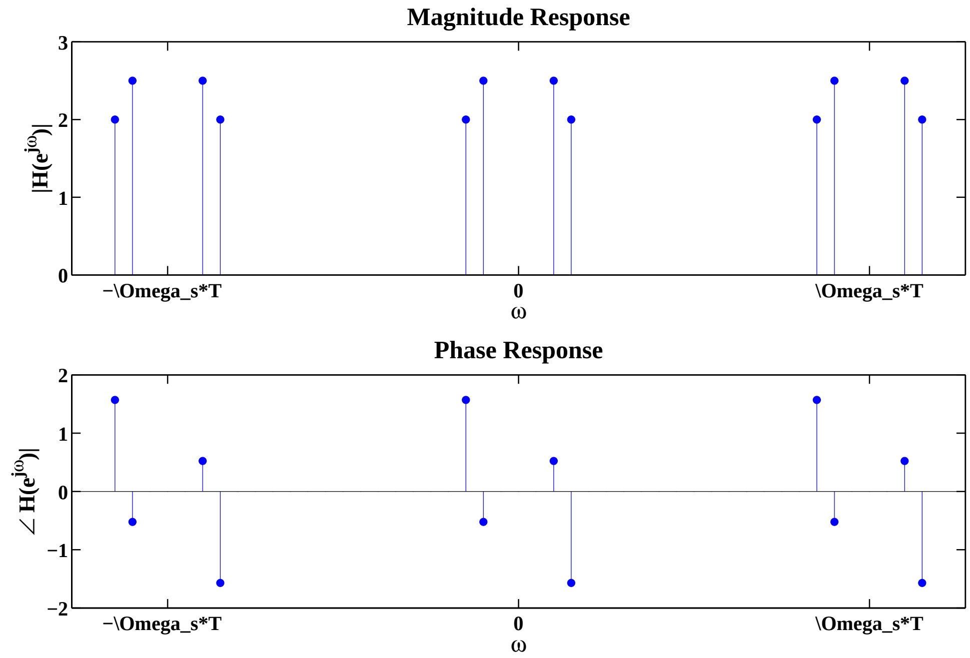
(d) tba.

MATLAB script:

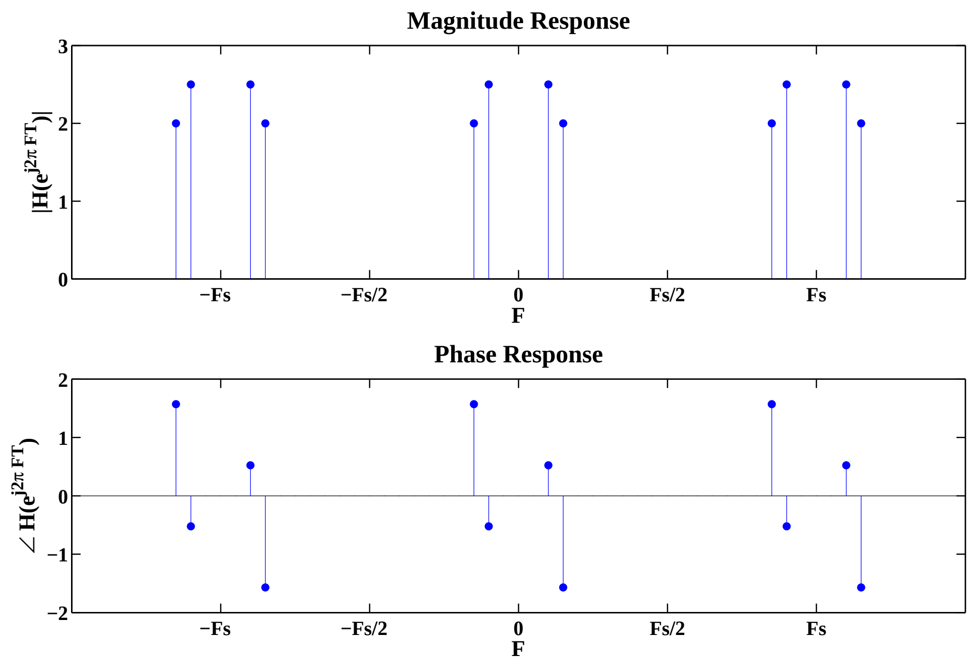
```
% P0601: Illustrates the alias distortion
close all; clc
Fs = 1e3; % Part (a)
% Fs = 500; % Part (b)
% Fs = 100; % Part (c)
T = 1/Fs;
FH = 150;
FL = FH+Fs;
F = -FL:50:FL;
X = zeros(1,length(F));
for k = -1:1;
ind = F == -150+k*Fs; X(ind) = X(ind)-2/j;
ind = F == -100+k*Fs; X(ind) = X(ind)+5/2*exp(-j*pi/6);
ind = F == 100+k*Fs; X(ind) = X(ind)+5/2*exp(j*pi/6);
ind = F == 150+k*Fs; X(ind) = X(ind)+2/j;
end
ind = X==0;
X(ind) = nan;
%% Plot:
hfa = figconfg('P0601a');
subplot(211)
stem(F*2*pi*T,abs(X),'filled')
ylim([0 max(abs(X))+0.5])
set(gca,'XTick',[-Fs*2*pi*T 0 Fs*2*pi*T])
set(gca,'XTickLabel',{'-\Omega_s*T','0','\Omega_s*T'})
xlabel('\omega','fontsize',LFS)
ylabel('|H(e^{j\omega})|','fontsize',LFS)
title('Magnitude Response','fontsize',TFS)
subplot(212)
stem(F*2*pi*T,angle(X),'filled')
set(gca,'XTick',[-Fs*2*pi*T 0 Fs*2*pi*T])
set(gca,'XTickLabel',{'-\Omega_s*T','0','\Omega_s*T'})
xlabel('\omega','fontsize',LFS)
ylabel('\angle H(e^{j\omega})|','fontsize',LFS)
title('Phase Response','fontsize',TFS)

hfb = figconfg('P0601b');
```

```
subplot(211)
stem(F,abs(X),'filled')
ylim([0 max(abs(X))+0.5])
set(gca,'XTick',[-Fs -Fs/2 0 Fs/2 Fs])
set(gca,'XTickLabel',{'-Fs','-Fs/2','0','Fs/2','Fs'})
xlabel('F','fontsize',LFS)
ylabel('|H(e^{j2\pi FT})|','fontsize',LFS)
title('Magnitude Response','fontsize',TFS)
subplot(212)
stem(F,angle(X),'filled')
set(gca,'XTick',[-Fs -Fs/2 0 Fs/2 Fs])
set(gca,'XTickLabel',{'-Fs','-Fs/2','0','Fs/2','Fs'})
xlabel('F','fontsize',LFS)
ylabel('\angle H(e^{j2\pi FT})','fontsize',LFS)
title('Phase Response','fontsize',TFS)
```

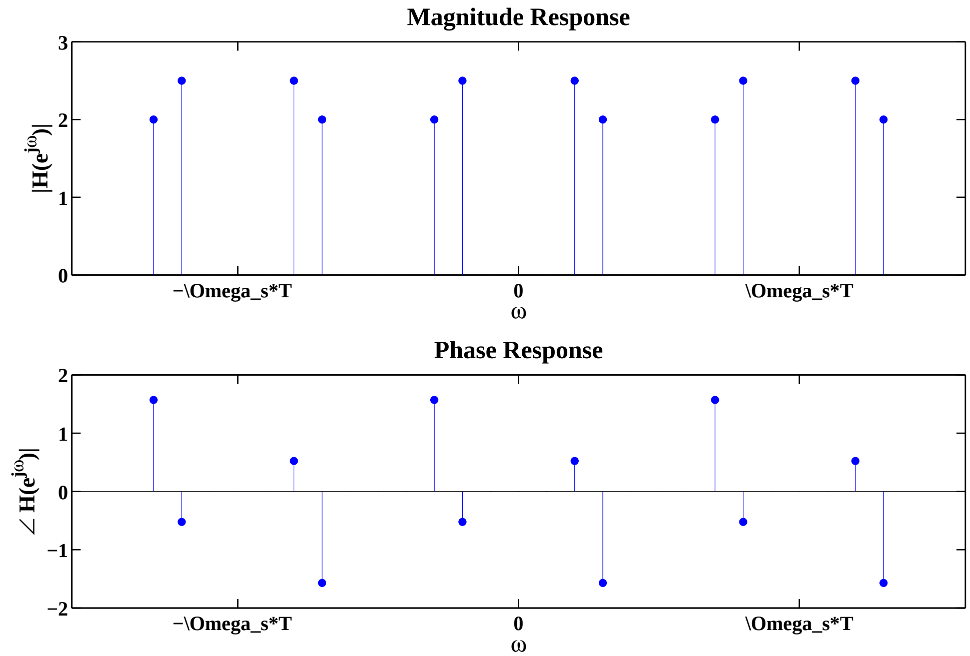


(a)

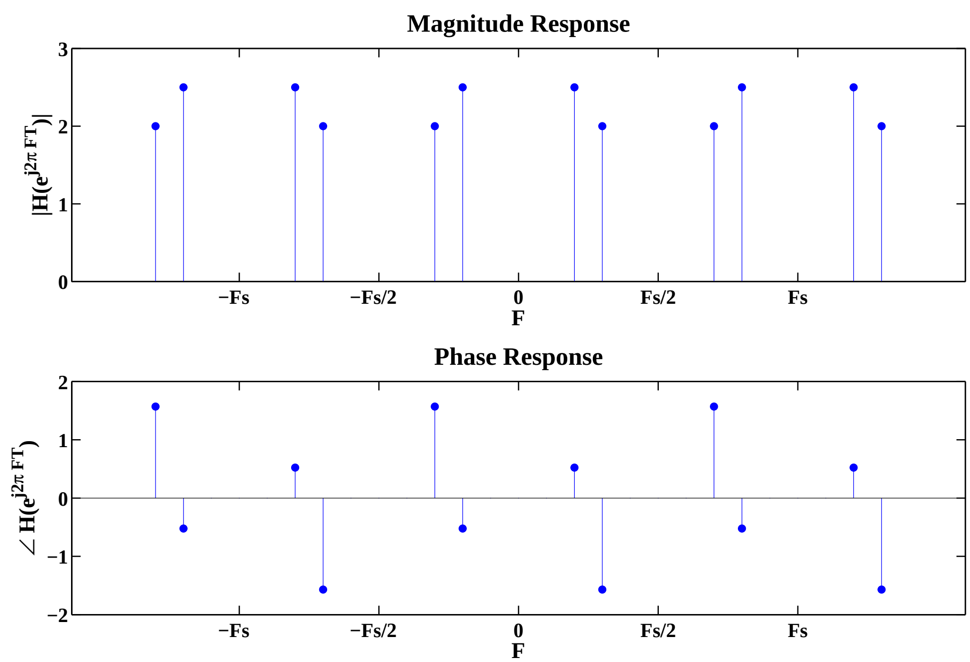


(b)

FIGURE 6.1: Spectra of $X\left(e^{j\omega}\right)$ as a function of (a) ω in $\frac{\text{rad}}{\text{sam}}$ and (b) F in Hz when the sample rate is $F_s = 1$ KHz.

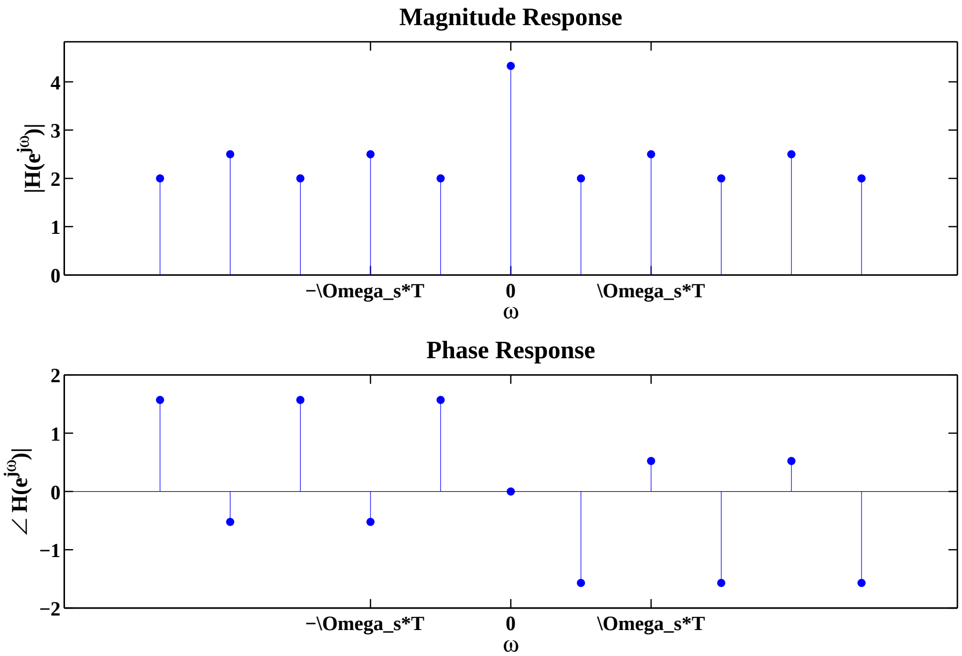


(a)

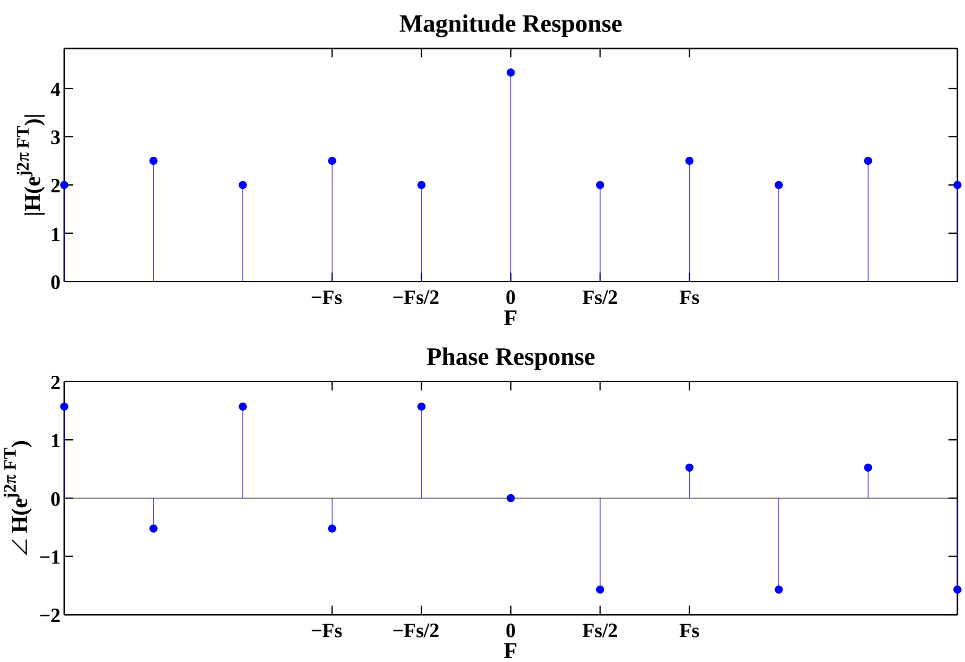


(b)

FIGURE 6.2: Spectra of $X(\mathrm{e}^{\mathrm{j}\omega})$ as a function of (a) ω in $\frac{\text{rad}}{\text{sam}}$ and (b) F in Hz when the sample rate is $F_s = 500$ Hz.



(a)



(b)

FIGURE 6.3: Spectra of $X(e^{j\omega})$ as a function of (a) ω in $\frac{\text{rad}}{\text{sam}}$ and (b) F in Hz when the sample rate is $F_s = 100$ Hz.

```
['Fs = ',num2str(Fs(3))],'location','best')
```

11. tba

12. (a) Solution:
    The quantizer resolution is:

$$\frac{10\text{v}}{2^8} = 0.0390625\text{v}$$

    (b) Solution:

$$\text{SQNR} = 10\log_{10}\text{SQNR} = 6.02B + 1.76 = 6.02\times 8 + 1.76 = 49.92\text{dB}$$

    (c) Solution:
    The sampling rate is:

$$F_s = \frac{2^{11}}{2^3} = 2^8\text{sam/sec}$$

    The folding frequency is $F_s/2 = 2^7$.
    The Nyquist rate is 500.

    (d) Solution:
    The reconstructed signal $y_c(t)$ is:

$$y_c(t) = 2\cos(200\pi t) - 3\sin(12\pi t)$$

13. Proof:
    (i) Linearity.

$$a_1 \cdot x_{\text{in1}}(nT) + a_2 \cdot x_{\text{in2}}(nT) = a_1 \cdot x_{\text{out1}}(t) + a_2 \cdot x_{\text{out2}}(t)$$

    The S&H system follows the superposition property, and hence is a linear system.
    (ii) Time-variance.

$$x_{\text{out}}(t-\tau) \neq x_{\text{out}}(t), \text{if } t-\tau \notin [nT, (n+1)T]$$

    Hence, the system is time-varying.

16. (a) See plot below.

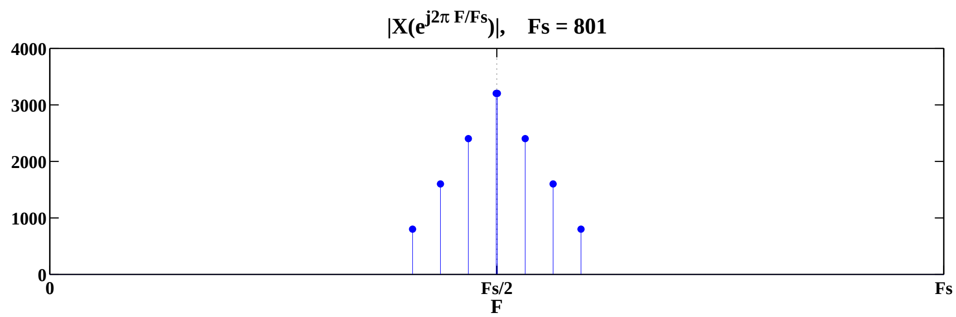


FIGURE 6.24: Spectrum of the sampled signal as a function of F Hz when the sampling rate is $F_s = 801$.

(b) See plot below.

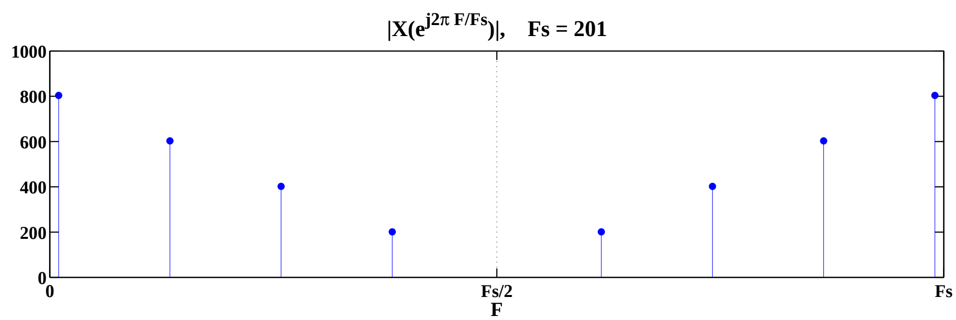


FIGURE 6.25: Spectrum of the sampled signal as a function of F Hz when the sampling rate is $F_s = 201$.

(c) tba.

```
% P0616: Sampling Illustration
close all; clc
%% Part (a):
% Fs = 801;
%% Part (b):
Fs = 201;
dF = 1;
F = 0:dF:Fs;
```

```
X = zeros(size(F));
FcosF = [325 350 375 400];
Fcos = [1 2 3 4];
while any(FcosF > Fs/2)
    ind = FcosF > Fs/2;
    FcosF(ind) = abs(FcosF(ind) - Fs);
end
for jj = -1:1
for ii = 1:length(FcosF)
    ind = abs(F)==abs(FcosF(ii)+jj*Fs);
    X(ind) = X(ind) + Fcos(ii)*Fs;
end
end
ind = X==0;
X(ind) = nan;
%% Plot:
hfa = figconfg('P0616a','long');
stem(F,abs(X),'filled');
xlim([0 Fs])
set(gca,'Xtick',[0 Fs/2 Fs])
set(gca,'Xticklabel',{'0','Fs/2','Fs'})
set(gca,'XGrid','on')
xlabel('F','fontsize',LFS)
title(['|X(e^{j2\pi F/Fs})|,    Fs = ',num2str(Fs)],'fontsize',TFS)
```

17. Solution:

$$F'_{\mathrm{L}} = 105 - 5 = 100\mathrm{Hz}, \qquad F'_{\mathrm{H}} = 145 + 5 = 150\mathrm{Hz}$$

The bandwidth is

$$B = F'_{\mathrm{H}} - F'_{\mathrm{L}} = 50\mathrm{Hz}$$

The minimum sampling rate is computed by

$$\min F_{\mathrm{s}} = 2F'_{\mathrm{H}}/\lfloor F'_{\mathrm{H}}/B \rfloor = 100\mathrm{Hz}$$

MATLAB script:

```
% P0617: Sampling Illustration
close all; clc
```

18. Proof:

$$p_{\rm c}(x,y)=\begin{cases}1/A^2, & |x|<A/2, \quad |y|<A/2\\ 0, & \text{otherwise}\end{cases} \tag{6.89}$$

$$P_{\rm c}(F_x,F_y)=\frac{\sin(\pi F_x A)}{\pi F_x A}\times\frac{\sin(\pi F_y A)}{\pi F_y A} \tag{6.90}$$

$$\begin{aligned}P_{\rm c}(F_x,F_y)&=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}p_{\rm c}(x,y)\mathrm{e}^{-\mathrm{j}2\pi(xF_x+yF_y)}\,\mathrm{d}x\mathrm{d}y\\ &=\int_{-\frac{A}{2}}^{\frac{A}{2}}\int_{-\frac{A}{2}}^{\frac{A}{2}}\frac{1}{A^2}\mathrm{e}^{-\mathrm{j}2\pi(xF_x+yF_y)}\,\mathrm{d}x\mathrm{d}y\\ &=\left(\int_{-\frac{A}{2}}^{\frac{A}{2}}\frac{1}{A}\mathrm{e}^{-\mathrm{j}2\pi xF_x}\,\mathrm{d}x\right)\cdot\left(\int_{-\frac{A}{2}}^{\frac{A}{2}}\frac{1}{A}\mathrm{e}^{-\mathrm{j}2\pi yF_y}\,\mathrm{d}y\right)\\ &=\frac{\mathrm{e}^{-\mathrm{j}2\pi xF_x}\Big|_{-\frac{A}{2}}^{\frac{A}{2}}}{A\cdot(-\mathrm{j}2\pi F_x)}\cdot\frac{\mathrm{e}^{-\mathrm{j}2\pi yF_y}\Big|_{-\frac{A}{2}}^{\frac{A}{2}}}{A\cdot(-\mathrm{j}2\pi F_y)}\\ &=\frac{\sin(\pi F_x A)}{\pi F_x A}\times\frac{\sin(\pi F_y A)}{\pi F_y A}\end{aligned}$$

19. (a) Solution:

$$s_{\rm c}(x,y)=3\cos(2.4\pi x+2.6\pi y)=3\cos(2.4\pi x)\cos(2.6\pi y)-3\sin(2.4\pi x)\sin(2.6\pi y)$$

$$s[m,n]=3\cos(0.8\pi m+1.3\pi n)$$

$$s_{\rm r}(x,y)=3\cos(1.6\pi x-2.6\pi y)$$

(b) Solution:

$$s[m,n]=3\cos(1.2\pi m+0.8667\pi n)$$

$$s_{\rm r}(x,y)=3\cos(2.4\pi x-1.4\pi y)$$

(c) Solution:

$$s[m,n]=3\cos(0.8\pi m+0.8667\pi n)$$

$$s_{\rm r}(x,y)=3\cos(2.4\pi x+2.6\pi y)$$

20. (a) Solution:

$$s_{\rm fa}[m,n]=\frac{1}{\Delta x\Delta y}\int_{m\Delta x-\frac{\Delta x}{2}}^{m\Delta x+\frac{\Delta x}{2}}\int_{n\Delta y-\frac{\Delta y}{2}}^{n\Delta y+\frac{\Delta y}{2}}s_{\rm c}(x,y)\mathrm{d}x\mathrm{d}y$$

(b) tba

(c) tba

Basic Problems

21. Proof:
The sampler is:

$$x_{\text{out}}(t) = x_{\text{in}}(nT); \quad nT \leq t < (n+1)T, \ \forall n$$

(i) Memoryless. The current system value is only related to the current time index and is not affected by previous system values. Hence, the sampler is memoryless.
(ii) Linearity.

$$a_1 \cdot x_{\text{in1}}(nT) + a_2 \cdot x_{\text{in2}}(nT) = a_1 \cdot x_{\text{out1}}(t) + a_2 \cdot x_{\text{out2}}(t)$$

The S&H system follows the superposition property, and hence is a linear system.
(iii) Time-variance.

$$x_{\text{out}}(t-\tau) \neq x_{\text{out}}(t), \text{if } t-\tau \notin [nT, (n+1)T]$$

Hence, the system is time-varying.

22. Solution:
The spectra of the continuous signal $x_{\text{c}}(t)$ is

$$X_{\text{c}}(\text{j}2\pi F) = \begin{cases} 3, & F = 0 \\ \text{j}, & F = 8 \\ -\text{j}, & F = -8 \\ 5, & F = 12 \\ 5, & F = -12 \\ 0, & \text{otherwise} \end{cases}$$

The spectra of sampled sequence $x[n]$ is:

$$X(\text{e}^{\text{j}\omega})\big|_{\omega=2\pi F/F_{\text{s}}} = F_{\text{s}} \sum_{n=-\infty}^{\infty} X_{\text{c}}[\text{j}2\pi(F - nF_{\text{s}})]$$

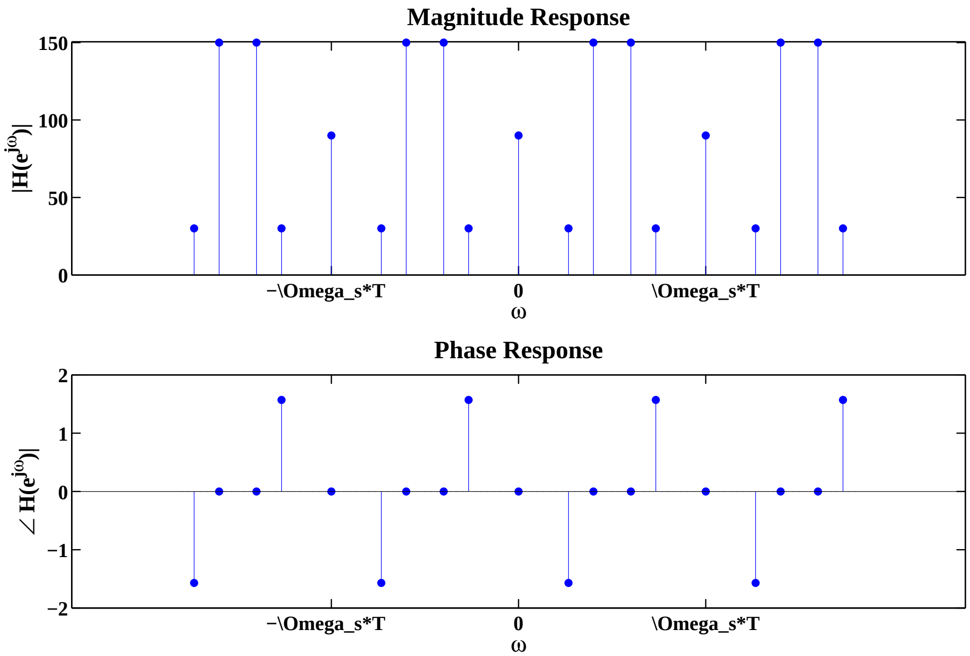
$x_{\text{c}}(t)$ can be recovered if (a) $F_{\text{s}} = 30$ Hz, and can NOT be recovered if (b) $F_{\text{s}} = 20$ Hz, (c) $F_{\text{s}} = 15$ Hz.

MATLAB script:

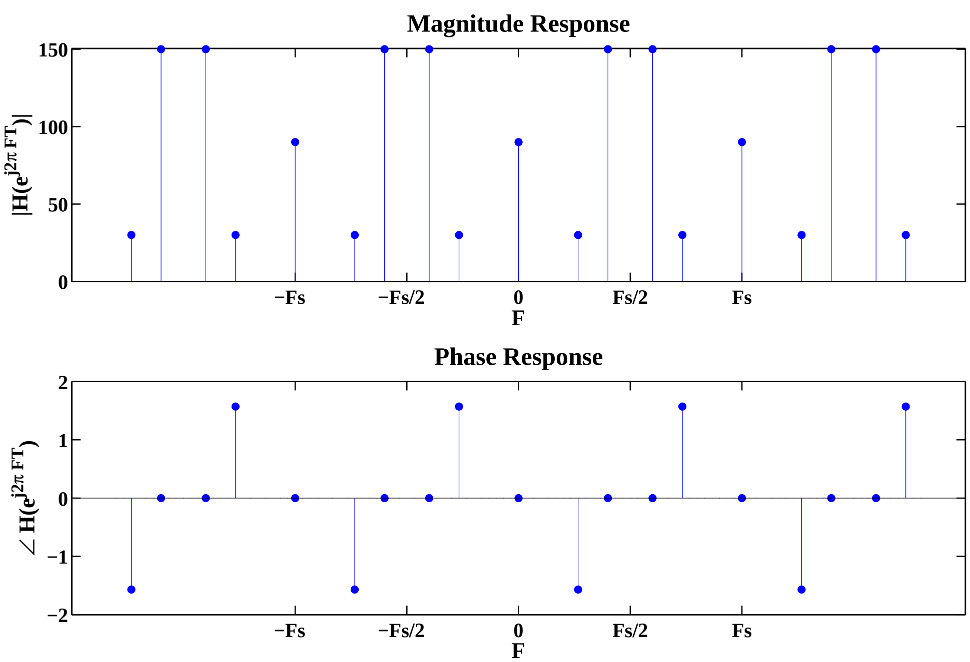
```
% P0622: Illustrates the alias distortion
close all; clc
Fs = 30; % Part (a)
% Fs = 20; % Part (b)
% Fs = 15; % Part (c)
T = 1/Fs;
FH = 24;
FL = FH+Fs;
F = -FL:1:FL;
X = zeros(1,length(F));
for k = -5:5;
ind = F == k*Fs; X(ind) = X(ind)+3*Fs;
ind = F == -12+k*Fs; X(ind) = X(ind)+5*Fs;
ind = F == -8+k*Fs; X(ind) = X(ind)-1/j*Fs;
ind = F == 8+k*Fs; X(ind) = X(ind)+1/j*Fs;
ind = F == 12+k*Fs; X(ind) = X(ind)+5*Fs;
end
ind = X==0;
X(ind) = nan;
%% Plot:
hfa = figconfg('P0622a');
subplot(211)
stem(F*2*pi*T,abs(X),'filled')
ylim([0 max(abs(X))+0.5])
set(gca,'XTick',[-Fs*2*pi*T 0 Fs*2*pi*T])
set(gca,'XTickLabel',{'-\Omega_s*T','0','\Omega_s*T'})
xlabel('\omega','fontsize',LFS)
ylabel('|H(e^{j\omega})|','fontsize',LFS)
title('Magnitude Response','fontsize',TFS)
subplot(212)
stem(F*2*pi*T,angle(X),'filled')
set(gca,'XTick',[-Fs*2*pi*T 0 Fs*2*pi*T])
set(gca,'XTickLabel',{'-\Omega_s*T','0','\Omega_s*T'})
xlabel('\omega','fontsize',LFS)
ylabel('\angle H(e^{j\omega})|','fontsize',LFS)
title('Phase Response','fontsize',TFS)

hfb = figconfg('P0622b');
subplot(211)
stem(F,abs(X),'filled')
```

```
ylim([0 max(abs(X))+0.5])
set(gca,'XTick',[-Fs -Fs/2 0 Fs/2 Fs])
set(gca,'XTickLabel',{'-Fs','-Fs/2','0','Fs/2','Fs'})
xlabel('F','fontsize',LFS)
ylabel('|H(e^{j2\pi FT})|','fontsize',LFS)
title('Magnitude Response','fontsize',TFS)
subplot(212)
stem(F,angle(X),'filled')
set(gca,'XTick',[-Fs -Fs/2 0 Fs/2 Fs])
set(gca,'XTickLabel',{'-Fs','-Fs/2','0','Fs/2','Fs'})
xlabel('F','fontsize',LFS)
ylabel('\angle H(e^{j2\pi FT})','fontsize',LFS)
title('Phase Response','fontsize',TFS)
```

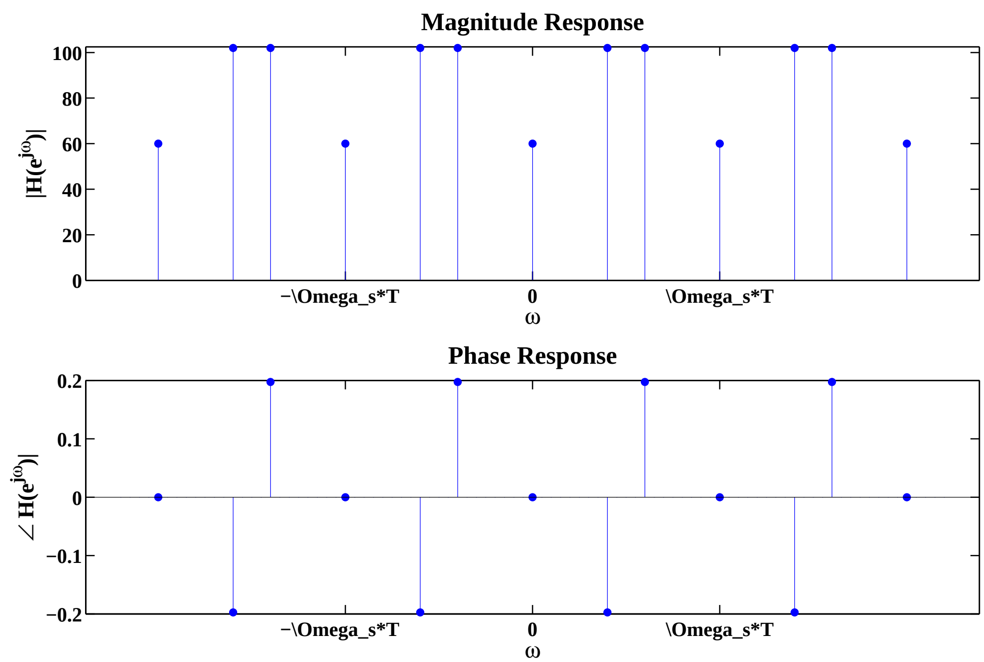


(a)

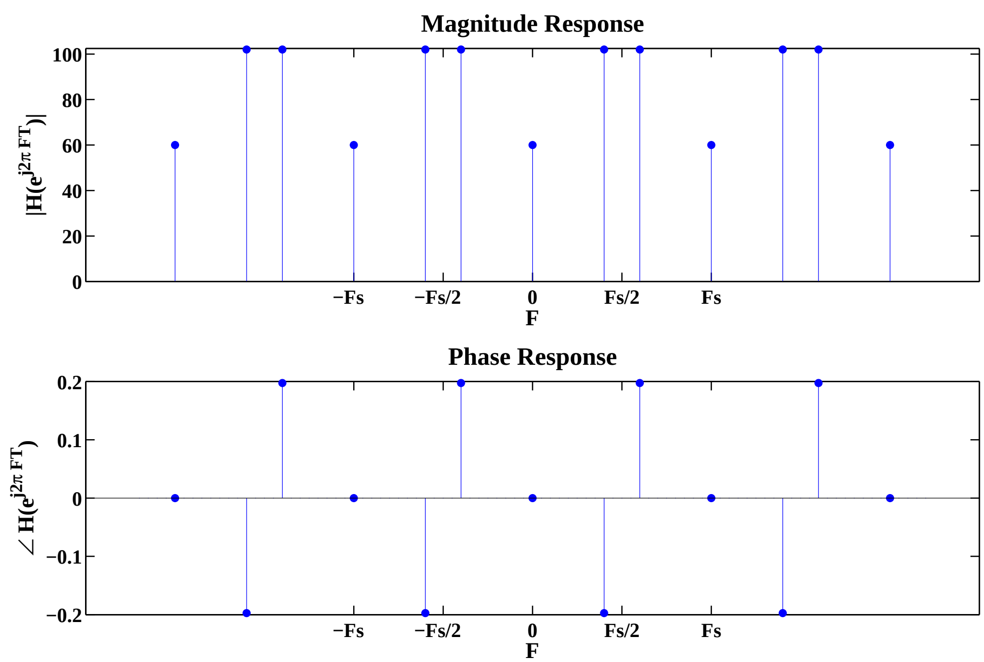


(b)

FIGURE 6.27: Spectra of $X(e^{j\omega})$ as a function of (a) ω in $\frac{\text{rad}}{\text{sam}}$ and (b) F in Hz when the sample rate is $F_s = 30$ KHz.

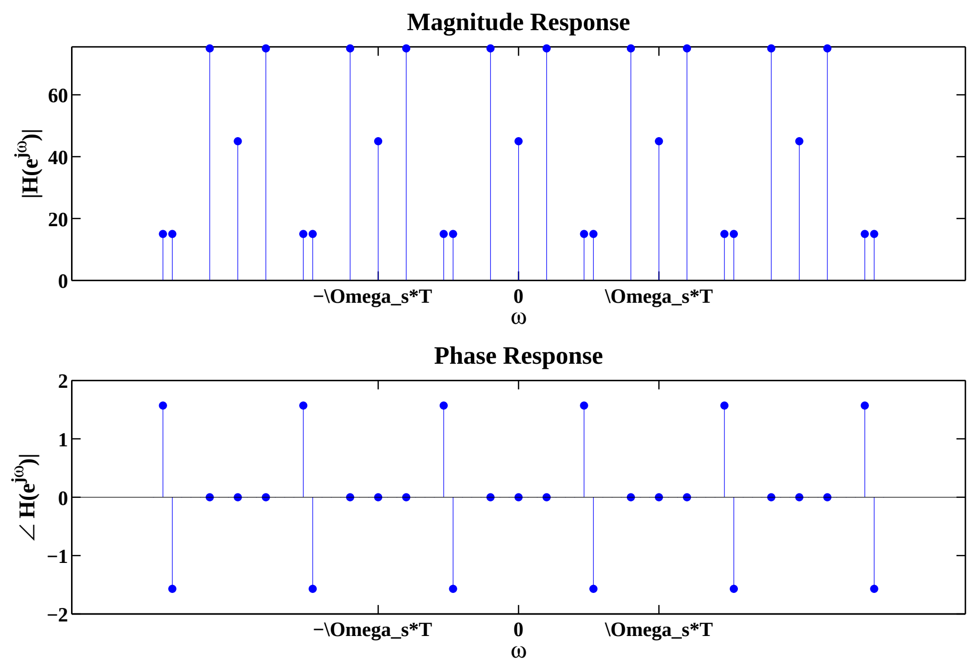


(a)

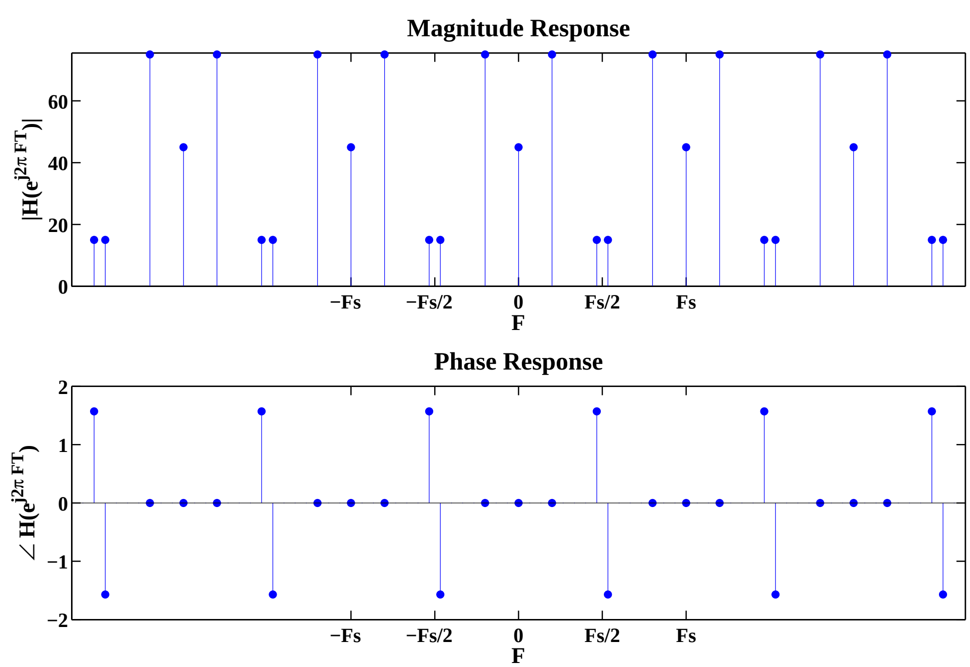


(b)

FIGURE 6.28: Spectra of $X(e^{j\omega})$ as a function of (a) ω in $\frac{\text{rad}}{\text{sam}}$ and (b) F in Hz when the sample rate is $F_s = 20$ KHz.



(a)



(b)

FIGURE 6.29: Spectra of $X(\mathrm{e}^{\mathrm{j}\omega})$ as a function of (a) ω in $\frac{\text{rad}}{\text{sam}}$ and (b) F in Hz when the sample rate is $F_s = 15$ KHz.

```
while F2y > Fs/2
    F2y = F2y-Fs;
end
yr = 3*cos(2*pi*F1y*t+pi/4)+3*sin(2*pi*F2y*t);
%% Plot:
hfa = figconfg('P0626a','long');
plot(t,xc); hold on
plot(nT,xn,'.','color','red')
xlabel('t','fontsize',LFS)
title(['F_1 = ',num2str(F1),',  F_2 = ',num2str(F2)],'fontsize',TFS)
legend('x_c(t)','x[n]','location','northeast')
hfb = figconfg('P0626b','long');
plot(t,xc,t,yr,nT,xn,'.')
xlabel('t','fontsize',LFS)
title(['F_1 = ',num2str(F1),',  F_2 = ',num2str(F2)],'fontsize',TFS)
legend('x_c(t)','y_r(t)','x[n]','location','northeast')
```

27. (a) Solution:

$$B = \frac{64\text{k}}{8\text{k}} = 8\text{bits/sam}$$

The quantizer step is:

$$\frac{10\text{v}}{2^8} = 0.0390625\text{v}$$

(b) Solution:
The SQNR is:

$$\text{SQNR} = 10\log_{10}\text{SQNR} = 6.02B+1.76 = 6.02*B+1.76 = 49.92\text{dB}$$

(c) the folding frequency
Solution:
The folding frequency is $F_s/2 = 4\text{k}$.

(d) Solution:
The reconstructed signal $x_r(t)$ is:

$$x_r(t) = -5\sin[6000\pi t - \pi/2]$$

28. Solution:

$$B = F_H - F_L = 20 - 18.1 = 1.9\text{KHz}$$

35. The same as P0623

36. Solution:
The spectra of the sampled sequence $x[n]$ is:

$$X(e^{j\omega})\big|_{\omega=\Omega/F_s} = F_s \sum_{k=-\infty}^{\infty} X_c[j(\Omega - 2\pi k F_s)]$$

The signal $x_c(t)$ can NOT be recovered when the sampling interval is (a) $T = \pi$, (b) $T = 0.5\pi$, (c) $T = 0.2\pi$.

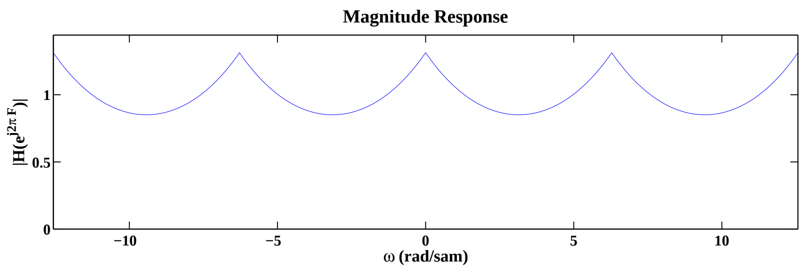


FIGURE 6.51: Magnitude response of $X(e^{j\omega})$ as a function of ω in $\frac{\text{rad}}{\text{sam}}$ when the sampling interval is $T = \pi$.

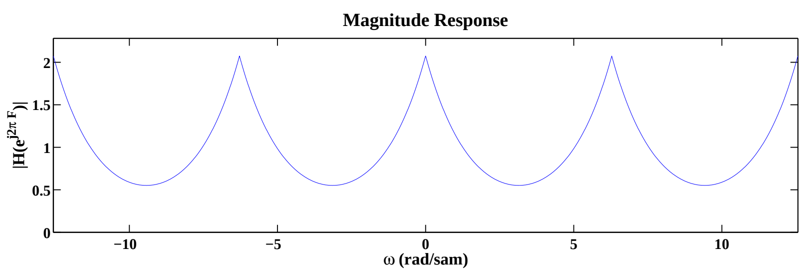


FIGURE 6.52: Magnitude response of $X(e^{j\omega})$ as a function of ω in $\frac{\text{rad}}{\text{sam}}$ when the sampling interval is $T = 0.5\pi$.

MATLAB script:

```
% P0636: Plot the spectra of sampled sequence
close all; clc
T = pi; % Part a
% T = 0.5*pi; % Part b
```

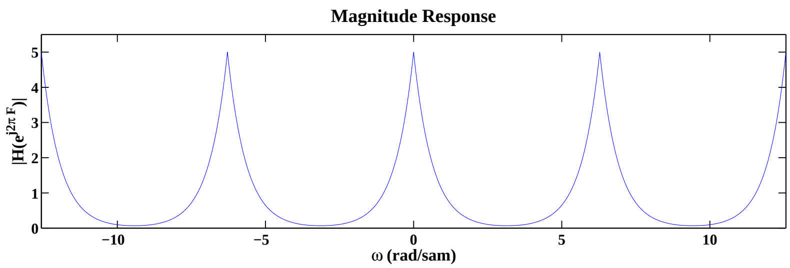


FIGURE 6.53: Magnitude response of $X(e^{j\omega})$ as a function of ω in $\frac{\text{rad}}{\text{sam}}$ when the sampling interval is $T = 0.2\pi$.

```
% T = 0.2*pi; % Part c
Fs = 1/T;
Omegas = 2*pi/T;
Omega = -2*Omegas:0.01:2*Omegas;
X = zeros(size(Omega));
for k = -10:1:10
X = X + pi*exp(-abs(Omega-k*Omegas))/T;
end
%% Plot:
hfa = figconfg('P0636a','long');
plot(Omega*T,abs(X))
xlabel('\omega (rad/sam)','fontsize',LFS)
ylabel('|H(e^{j2\pi F})|','fontsize',LFS)
ylim([0 max(abs(X))*1.1])
xlim([-2*Omegas*T 2*Omegas*T])
title('Magnitude Response','fontsize',TFS)
```

37. Solution:

The spectra of the continuous signal $x_c(t)$ is:

$$X_c(j2\pi F) = \begin{cases} 1 - \frac{F^2}{25}, & |F| \le 5 \\ 0, & \text{otherwise} \end{cases}$$

The spectra of the sampled sequence $x[n]$ is:

$$X(e^{j\omega})\big|_{\omega=2\pi F/F_s} = F_s \sum_{k=-\infty}^{\infty} X_c[j2\pi(F - kF_s)]$$