

EXAM No. 1

Problem No. 1: Consider the LTCC difference equation:

$$y[n] = a_1y[n - 1] + a_2y[n - 2] + x[n] + b_1x[n - 1]$$

where all coefficients are real numbers.

(a) Derive the conditions under which the system is stable.

$$\begin{aligned} y[n] &= a_1y[n - 1] + a_2y[n - 2] + x[n] + b_1x[n - 1] \\ Y(z) &= a_1Y(z)z^{-1} + a_2Y(z)z^{-2} + X(z) + b_1X(z)z^{-1} \end{aligned} \quad (1)$$

$$\begin{aligned} Y(z) - a_1Y(z)z^{-1} - a_2Y(z)z^{-2} &= X(z) + b_1X(z)z^{-1} \\ Y(z)[1 - a_1z^{-1} - a_2z^{-2}] &= X(z)[1 + b_1z^{-1}] \end{aligned} \quad (2)$$

$$\begin{aligned} H(z) = \frac{Y(z)}{X(z)} &= \frac{1 + b_1z^{-1}}{1 - a_1z^{-1} - a_2z^{-2}} \\ &= \frac{1 + b_1z^{-1}}{(1 - p_1z^{-1})(1 - p_2z^{-1})} \\ &= \frac{1 + b_1z^{-1}}{1 - \underbrace{2\operatorname{Re}\{p_1\}}_{a_1}z^{-1} + \underbrace{|p_1|^2}_{a_2}z^{-2}} \end{aligned} \quad (3)$$

The system is stable if the poles p_1 and p_2 are less than 1 or inside the unit circle. The system is also stable if a_2 is less than 1. If the poles are equal to 1 or directly on the unit circle then the system is marginally stable. The numerator or zero does not contribute to the stability of the system.

(b) Assume $b_1 = 0$. Derive an expression for the impulse response and explain the role a_1 and a_2 play in the shape of this impulse response. Be as specific as possible. For example, if the signal oscillates, write an equation for the frequency of oscillation in terms of the coefficients.

$$H(z) \Big|_{z=e^{j\omega}} = \frac{1}{1 - a_1z^{-1} - a_2z^{-2}} \Big|_{z=e^{j\omega}} \quad (4)$$

$$H(e^{j\omega}) = \frac{1}{1 - a_1e^{-j\omega} - a_2e^{-2j\omega}} \quad (5)$$

As $\omega \rightarrow \infty$ the exponential terms approaches 0. Therefore the impulse response is a damped sinusoid.

(c) Assume the system is stable and $b_1 \neq 0$. Plot the location of the poles and zeroes in the z-plane and explain how these influence the frequency response. (Hint: describe the relationship between the z-transform and the frequency response.)

$$p_{1,2} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2} \quad (6)$$

As we evaluate $H(z)$ around the unit circle the poles result in valleys or peaks, while zeros result in dips. This is for the magnitude. If we move the evaluation point closer to the origin, then the poles and zeros have a larger impact on the magnitude. However the peaks and dips will be much narrower.

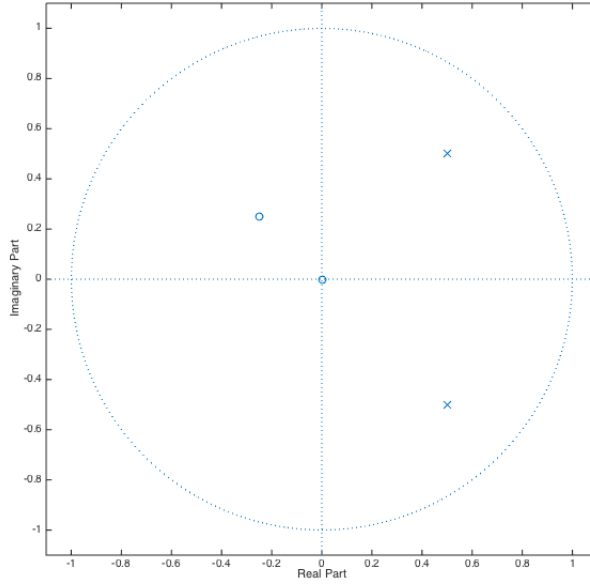


Figure 1: Z plane with poles at $0.5 \pm 0.5j$ and zeros at 0 and $-0.25 + 0.25j$.

Problem No. 2: Given the system shown:

(a) Find the impulse response, $h[n]$.

$$y[n] = x[n] + \frac{1}{2}x[n-1] + \frac{1}{8}x[n-2] \quad (7)$$

$$y[n] = x[n] \left(1 + \frac{1}{2}\delta[n-1] + \frac{1}{8}\delta[n-2] \right)$$

$$\boxed{h[n] = \frac{y[n]}{x[n]} = 1 + \frac{1}{2}\delta[n-1] + \frac{1}{8}\delta[n-2]} \quad (8)$$

(b) Find the transfer function, $H[z]$.

$$Y(z) = X(z) + \frac{1}{2}X(z)z^{-1} + \frac{1}{8}X(z)z^{-2} \quad (9)$$

$$Y(z) = X(z) \left[1 + \frac{1}{2}z^{-1} + \frac{1}{8}z^{-2} \right]$$

$$\boxed{H(z) = \frac{Y(z)}{X(z)} = 1 + \frac{1}{2}z^{-1} + \frac{1}{8}z^{-2}} \quad (10)$$

Problem No. 3: Given the signal $x[n] = [0 \ 1 \ 0 \ 2]$ represents a continuous-time signal sampled at 2 Hz.

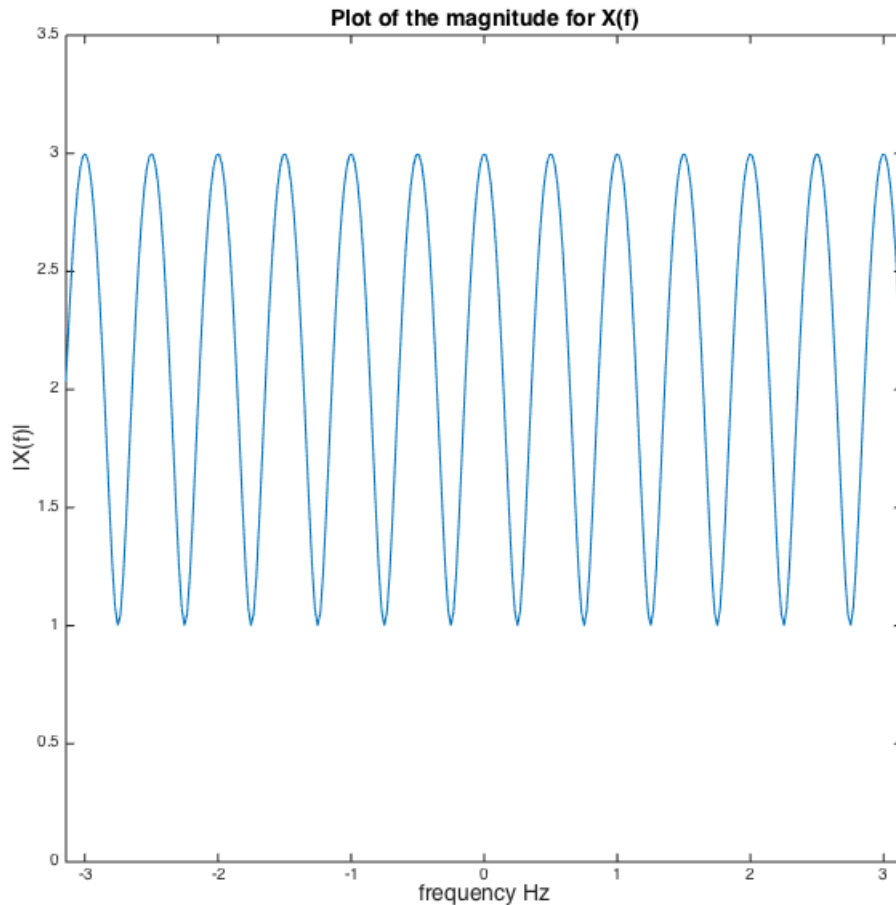
(a) Find the frequency response, $X(f)$, and sketch the magnitude as a function of frequency in Hz.

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} x[n]z^{-n} \\ &= (0)z^0 + (1)z^{-1} + (0)z^{-2} + (2)z^{-3} \\ &= z^{-1} + 2z^{-3} \end{aligned} \quad (11)$$

$$X(e^{j\omega}) = e^{-j\omega} + 2e^{-3j\omega} \quad (12)$$

$$\omega = 2\pi \frac{f}{f_s}$$

$$X(f) = e^{-j\pi f} + 2e^{-3j\pi f} \quad (13)$$



(b) Next, assume the signal is periodic with a period of 10 samples. Sketch the frequency response and explain the relationship between this answer and the answer to (a).

$$x[n] = [0 \ 1 \ 0 \ 2 \ 0 \ 0 \ 0 \ 0 \ 0]$$

The relationship between the answer in problem 3 a is that they are both a sinusoid function. However the frequency response for 3b is discrete because a periodic signal in the time domain yields a line spectra in the frequency domain.

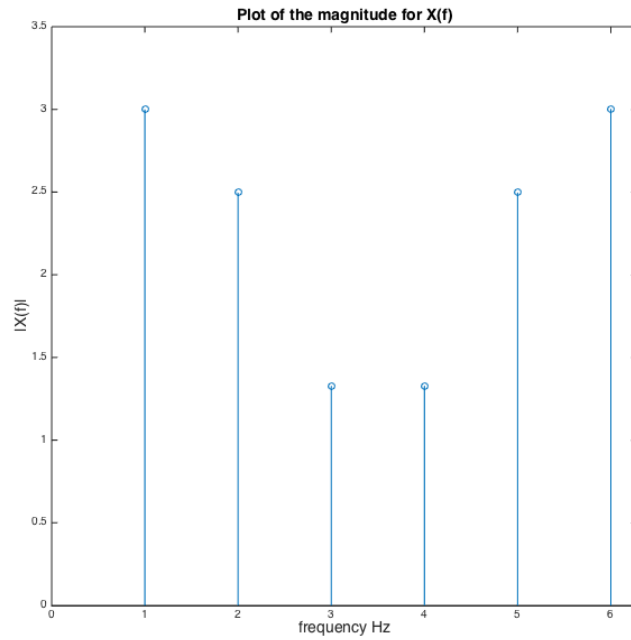


Figure 2: Frequency response for $x[n]$ periodic with 10 samples.

(c) How would your answer to (b) change if the period was 5 samples rather than 10 samples?

The frequency response will be much wider than that with 10 samples. This happens because a signal that is narrow in the time domain is wide in the frequency domain.

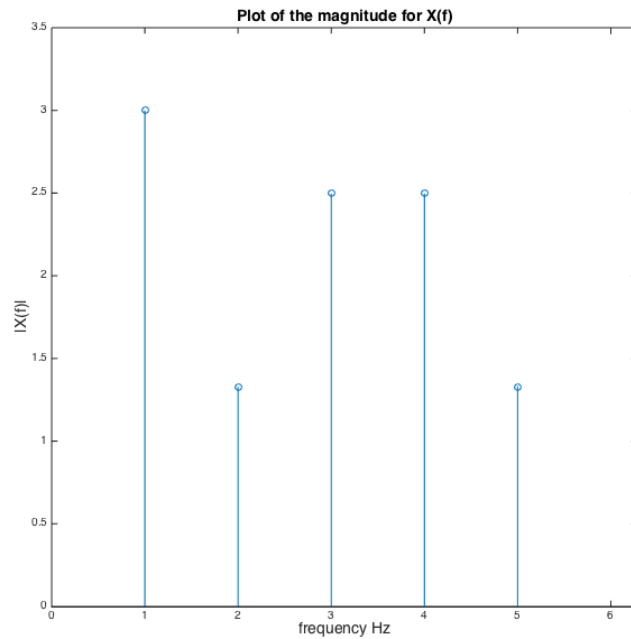


Figure 3: Frequency response with for $x[n]$ periodic with 5 samples.

Problem No. 4: The impulse response to a linear time-invariant system is given by:

(a) Draw a block diagram realization of the system (e.g, use delay lines, adders, and amplifiers).

$$h[n] = \begin{cases} 1 & n = -2 \\ 2 & n = -1 \\ 0 & n = 0 \\ -1 & n = 1 \\ -2 & n = 2 \\ 0 & \text{otherwise} \end{cases} \quad (14)$$

The original system is non-causal. In order to implement this system in Simulink, I had to shift the samples by 2.

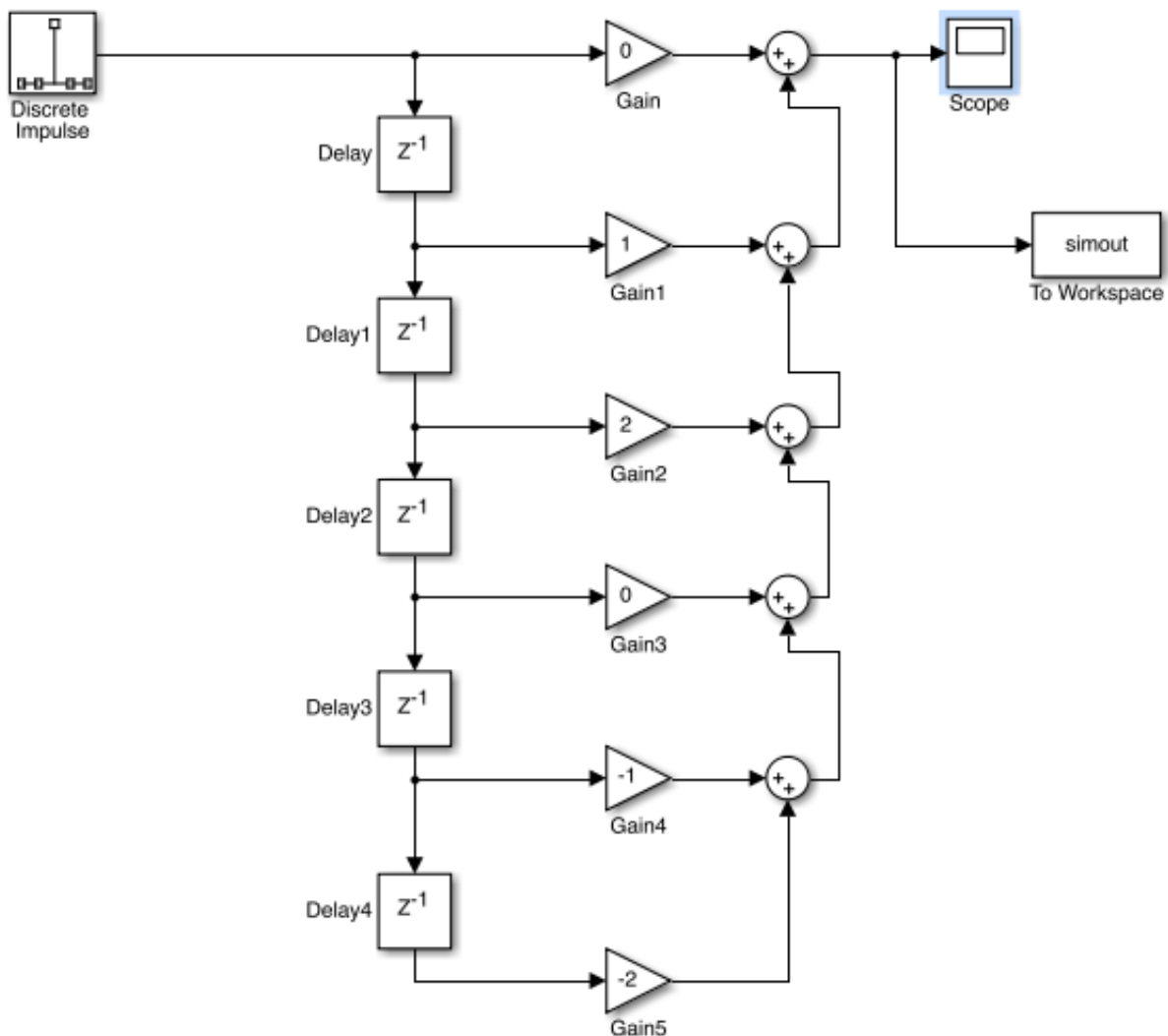


Figure 4: Block diagram from Simulink for $h[n]$.

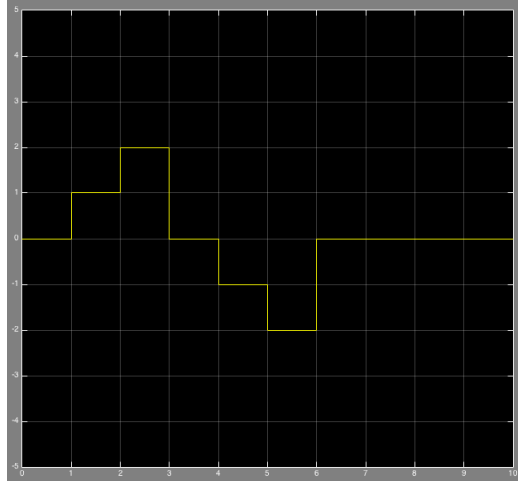
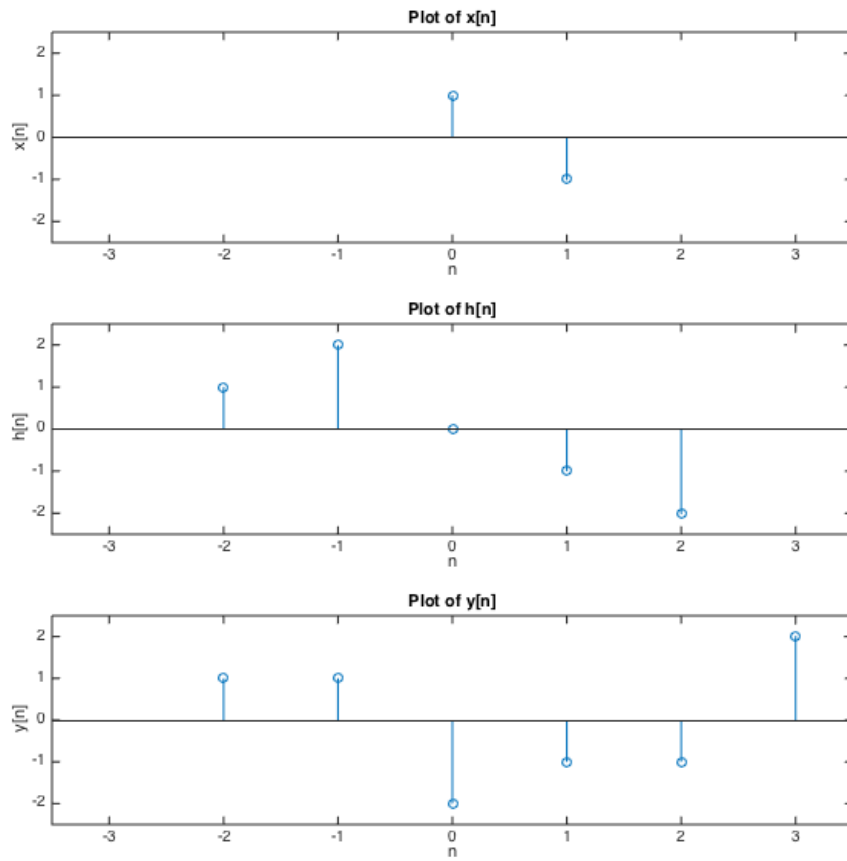


Figure 5: Signal from the scope.

(b) Suppose the input to this system is $x[n] = [1 \ -1]$. Compute the output.

$$y[n] = x[n] * h[n] \tag{15}$$



MATLAB script

```
%% Problem 3 (a)
n= 1000;
fs = 2;
f = linspace(0,2*pi,n);
w = 2*pi*f/fs;
X = exp(1i*w) + 2*exp(-3i*w);
figure(1);
plot(f,abs(X));
axis([0,2*pi,0,3.5]);
xlabel('frequency Hz','FontSize',14);
ylabel('|X(f)|','FontSize',14);
title('Plot of the magnitude for X(f)','FontSize',14);
%% Problem 3 (b)
x = [0 1 0 2 0 0 0 0 0 0];
X = fft(x);
figure(3);
stem(abs(X));
axis([0,2*pi,0,3.5]);
xlabel('frequency Hz','FontSize',14);
ylabel('|X(f)|','FontSize',14);
title('Plot of the magnitude for X(f)','FontSize',14);
%% Problem 3 (c)
x = [0 1 0 2 0];
X = fft(x);
figure(4);
stem(abs(X));
axis([0,2*pi,0,3.5]);
xlabel('frequency Hz','FontSize',14);
ylabel('|X(f)|','FontSize',14);
title('Plot of the magnitude for X(f)','FontSize',14);
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Problem 4 (b)
t = -2:1:2;
h = [ 1 2 0 -1 -2];
x = [1 -1];
y = conv(x,h)
subplot(3,1,1)
stem(t(3:4),x);
axis([-3.5,3.5,-2.5,2.5]);
xlabel('n'); ylabel('x[n]');
title('Plot of x[n]');
subplot(3,1,2);
stem(t,h);
axis([-3.5,3.5,-2.5,2.5]);
xlabel('n');ylabel('h[n]');
```

```
title('Plot of h[n]')
subplot(3,1,3)
stem([t 3],y)
axis([-3.5,3.5,-2.5,2.5])
xlabel('n');ylabel('y[n]');
title('Plot of y[n]')
```