Sinusoidal Response

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n)$$

$x(n) = e^{j\omega_0 n} \quad -\infty < n < \infty$

$$y(n) = \sum_m h(m)\, x(n-m)$$
$$= \sum_m h(m)\, e^{j\omega_0 (n-m)}$$
$$= e^{j\omega_0 n}\left[\sum_m h(m)\, e^{-j\omega_0 m}\right]$$
$$= H(\omega_0)\, \underbrace{e^{j\omega_0 n}}$$

where $H(\omega_0) = \sum_m h(m)\, e^{-j\omega_0 m}$

(1) sinewave in $\Leftrightarrow$ sinewave out

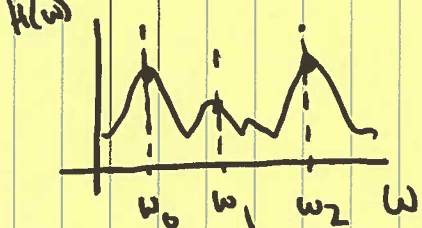
(2) $H(\omega_0) = |H(\omega_0)|\, e^{j\angle H(\omega_0)}$

$\therefore\ e^{j\omega_0 n} \xrightarrow{H} H(\omega_0)\, e^{j\omega_0 n}$

Impact:

Since all periodic signals can be represented as a sum of sinewaves (Fourier Series)

We can extend this result to all periodic signals.



$x(n) = e^{j\omega_0 n} + e^{j\omega_1 n} + e^{j\omega_2 n}$

$y(n) = H(\omega_0)\, e^{j\omega_0 n} + H(\omega_1)\, e^{j\omega_1 n} + H(\omega_2)\, e^{j\omega_2 n}$

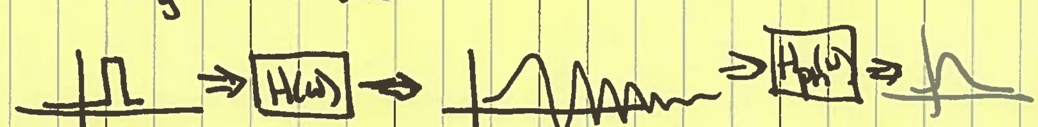
$$e^{j\omega_0 n} \xrightarrow{H} \underbrace{|H(\omega_0)|}_{\text{magnitude response}}\, e^{j\omega_0 n + j \overbrace{\arg H(\omega_0)}^{\text{phase response}}}$$

Phase delay:

$$d(\omega) = -\frac{\arg H(\omega)}{\omega}$$

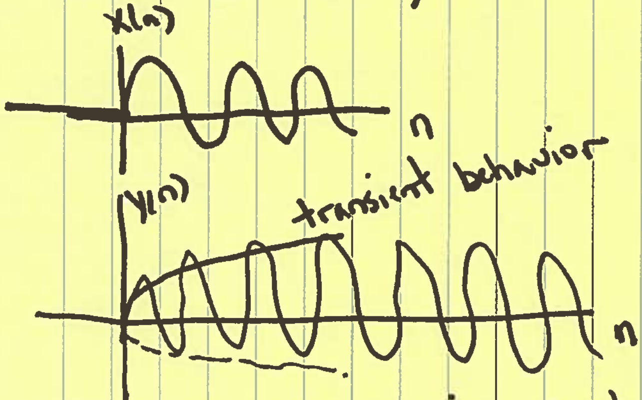
Group delay:

$$d_g(\omega) = -\frac{d}{d\omega} \arg H(\omega)$$



Transient Response

$x(n) = e^{j\omega_0 n} u(n)$



why is stability important?

transient will die down if stable

$$H(z) = \frac{N(z)}{(1-p_1 z^{-1})(1-p_2 z^{-1}) \cdots (1-p_m z^{-1})}$$

$$= \frac{A_1}{1-p_1 z^{-1}} + \frac{A_2}{1-p_2 z^{-1}} + \cdots$$

$$h(n) = B_1 (p_1)^n + B_2 (p_2)^n + \cdots$$

$$Y(z) = H(z) X(z)$$

$$y(n) = H(\omega_0) e^{j\omega_0 n} + B_1 (p_1)^n + B_2 (p_2)^n + \cdots$$

If the system is stable:

poles are inside the unit circle

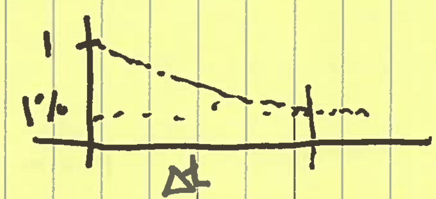
$|p_1| < 1$

$|p_2| < 1$

$\cdots$

$B_1 (p_1)^n \to 0$

$\therefore$ transient response decays and at steady-state we are left with $e^{j\omega_0 n}$

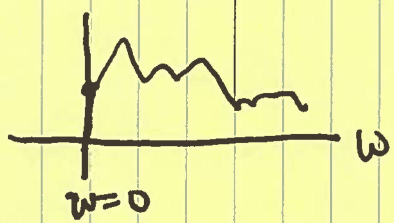


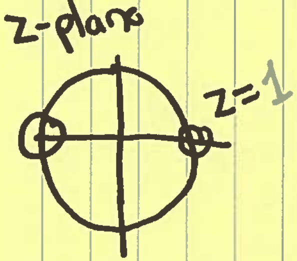
$$\rho^{n_{eff}} = \varepsilon$$

n_{eff} is the time it takes ρ^n to decay to some percentage of its initial value

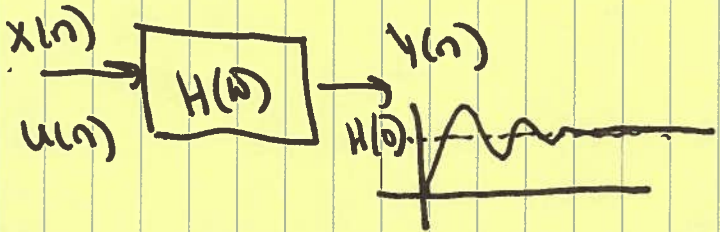
$$n_{eff} = \frac{\ln \varepsilon}{\ln \rho} \quad \text{(time constant)}$$

$X(n) = u(n)$





$H(0)$ is the DC value



$$H(0) = H(z)\Big|_{z=1} = \sum_{n=0}^{\infty} h(n) \quad \text{DC gain}$$

$$H(\pi) = H(z)\Big|_{z=-1} = \sum_{n=0}^{\infty} (-1)^n h(n) \quad \text{AC gain}$$