Lecture 23: DFT / FFT Algorithms

Recall:

$$T\hat{X}(f) = X(f) + X(f-f_s) + X(f+f_s) + \dots$$

The spectrum of $x(n)$:

$$\hat{X}(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

$$\hat{X}(f) = \sum_{n=-\infty}^{\infty} x(nT) e^{-j2\pi f nT} \qquad T = \frac{1}{f_s}$$

However, this is useless! Why?

$$\hat{X}_L(f) = \sum_{n=0}^{L-1} x(nT) e^{-j2\pi f nT}$$

Implications:

could work for periodic signals or time-limited signals

for other signals, like speech, distortion will occur

$$x_L(n) = x(n)\, w(n)$$

$$\text{where } w(n) = \begin{cases} 1 & 0 \le n \le L-1 \\ 0 & \text{elsewhere} \end{cases}$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

$$X_L(\omega) = \sum_{n=0}^{L-1} x(n) e^{-j\omega n}$$

multiplication in the time-domain is convolution in the freq. domain:

$$X_L(\omega) = \int_{-\pi}^{\pi} X(\omega') W(\omega-\omega') \frac{d\omega'}{2\pi}$$

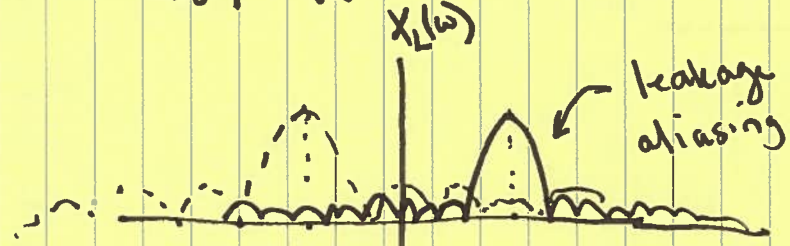
$$W(\omega) = \sum_{n=0}^{L-1} w(n) e^{-j\omega n}$$

$$W(z) = \sum_{n=0}^{L-1} w(n) z^{-n} = \sum_{n=0}^{L-1} z^{-n} = \frac{1-z^{-L}}{1-z^{-1}}$$

$$W(\omega) = \frac{1-e^{jL\omega}}{1-e^{-j\omega}}$$

Distortion occurs!

$x(n) = \sin \omega_0 n$



the "peak" in the spectrum shifts

"bias": the peak is at the wrong freq.

the distortion is due to time-domain windowing

(using a finite number of samples)

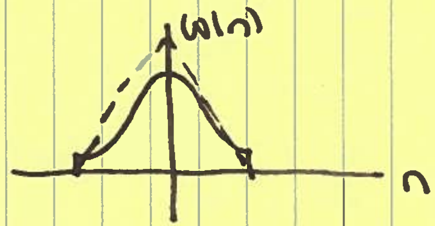
Can we "minimize" the leakage?

could make the window longer....

Hamming window:

$$w(n) = 0.54 - 0.46 \cos\left(\frac{2\pi n}{L-1}\right) \quad 0 \le n \le L-1$$

$$= 0 \quad \text{elsewhere}$$



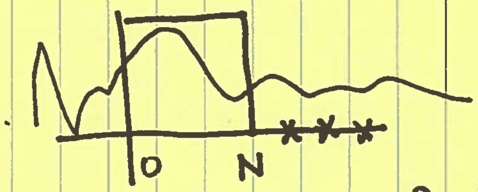
$$\hat{x}(n) = x(n)\, w(n)$$

Discrete Fourier Transform:

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi kn/N}$$

where $\frac{k}{N} f_s$ are frequency samples

we can interpolate the spectrum by zero-padding



add zeros to the end of the vector

$$\tilde{x}(n) = \begin{cases} x(n) & 0 \le n \le N-1 \\ 0 & N \le n \le M-1 \end{cases}$$

resolution is f_s/M where $M > N$

Summary:

$$X(f) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j2\pi f/f_s}$$

$$\rightarrow \tilde{X}(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi kn/N}$$