Lecture 11

Frequency Response

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

let $z = e^{j\omega}$

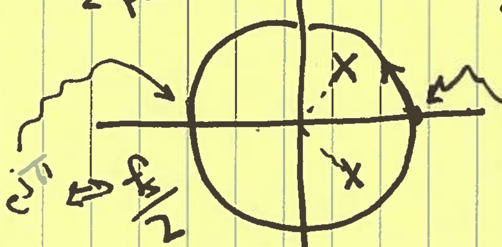
$$X(z)\Big|_{z=e^{j\omega}} = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} = X(\omega)$$

Notice: f_s (sample frequency) does not appear

ω is normalized frequency:

$$\omega = \frac{2\pi f}{f_s}$$

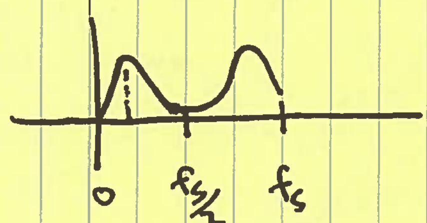
z-plane



$z = e^{j\omega}$

$|e^{j\omega}| = 1$

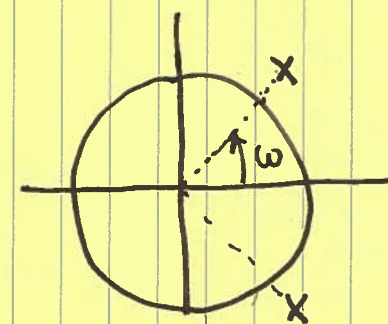
$\omega = 0, f_s, 2f_s$



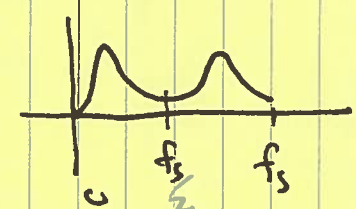
$$X(\omega)\Big|_{\omega=\frac{2\pi f}{f_s}} = X(f)$$

$$X(f) = \sum_{n=-\infty}^{\infty} x(nT) e^{-j2\pi f/f_s}$$

$$T = \frac{1}{f_s}$$



unstable system



Similarly,

$$x(n) \rightarrow \boxed{h(n)} \rightarrow y(n)$$

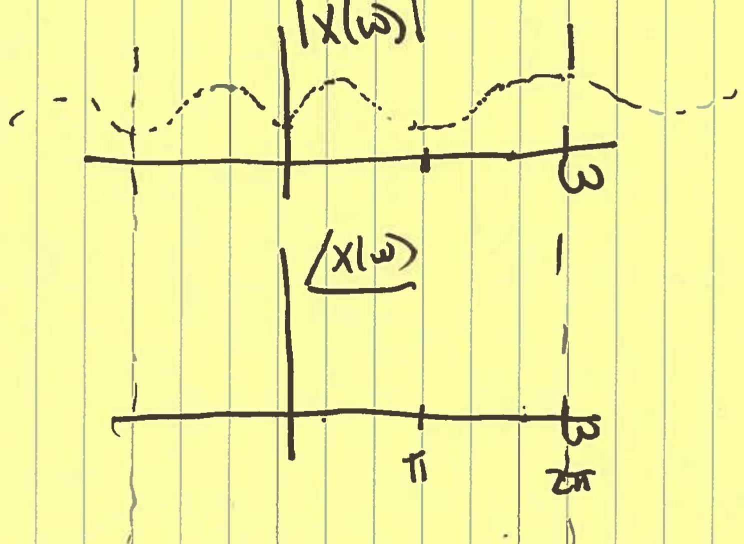
$$H(z)\Big|_{z=e^{j\omega}} = \sum_{n=-\infty}^{\infty} h(n) e^{-j\omega n}$$

$$x(n) = [\overset{n=0}{0}, -1, 3, 2]$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= (-1) z^{-1} + (3) z^{-2} + (2) z^{-3}$$

$$X(\omega) = X(z)\Big|_{z=e^{j\omega}} = (-1)(e^{-j\omega}) + (3) e^{-2j\omega} + (2) e^{-j3\omega}$$



periodic $\omega = 2\pi$

real signal mag. funct. is even

phase funct. is odd

Ex:

$$y(n) = a\,y(n-1) + x(n)$$

$$Y(z) = a z^{-1} Y(z) + X(z)$$

$$Y(z)[1 - a z^{-1}] = X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - a z^{-1}}$$

$$H(\omega) = H(z)\Big|_{z=e^{j\omega}} = \frac{1}{1 - a e^{-j\omega}}$$

Inverse Z-Transform

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega) e^{j\omega n}\, d\omega$$

Limit ourselves to:

$$X(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots}{a_0 + a_1 z^{-1} + \cdots}$$

Ratio of two polynomials

$$X(z) = \frac{A_1}{1+p_1 z^{-1}} + \frac{A_2}{1+p_2 z^{-1}} + \cdots$$

$$= A_1 (p_1)^n u(n) + A_2 (p_2)^n u(n) + \cdots$$

Ex: $X(z) = 1 + a_1 z^{-1}$

$$x(n) = \delta(n) + a_1 \delta(n-1)$$

Ex: $X(z) = a_1 z + a_2 + a_3 z^{-3}$

$$x(n) = a_1 \delta(n+1) + a_2 \delta(n) + a_3 \delta(n-3)$$

Ex: $X(z) = \dfrac{1+3z^{-1}}{2+7z^{-2}+9z^{-5}}$

$$= a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots$$

$$x(n) = a_0 \delta(n) + a_1 \delta(n-1) \cdots$$

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n)$$

$H(z) = 1 + a_1 z^{-1}$

$X(z) = 1 + b_1 z^{-2}$

$y(n) = ?$

$$Y(z) = H(z)X(z)$$
$$= (1 + a_1 z^{-1})(1 + b_1 z^{-2})$$
$$= 1 + a_1 z^{-1} + b_1 z^{-2} + a_1 b_1 z^{-3}$$

$$y(n) = \delta(n) + a_1\delta(n-1) + b_1\delta(n-2) + a_1 b_1 \delta(n-3)$$

$$h(n) = \delta(n) + a_1\delta(n-1)$$
$$x(n) = \delta(n) + b_1\delta(n-2)$$

$$y(n) = h(n) * x(n)$$

Remember:

$$Y(z) = X(z)H(z)$$

$$Y(\omega) = X(\omega)H(\omega)$$

Important Consequence:

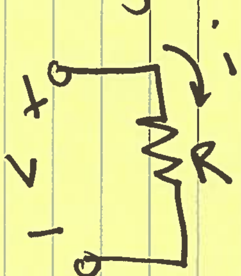
$$\sum_{n=-\infty}^{\infty} |x(n)|^2 \triangleq \text{Energy}$$

$$\sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega$$

Parseval's Theorem:

energy in time domain is equivalent to energy in the frequency domain

Analog:



$$E = \int v(t)\, i(t)\, dt$$

$$= \int |x(t)|^2 \, dt$$