CHAPTER 3

The z-Transform

Tutorial Problems

1. (a) Solution:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n (u[n] - u[n-10])z^{-n}$$
$$= \sum_{n=0}^{9} \left(\frac{1}{2}\right)^n z^{-n} = \frac{1 - \left(\frac{1}{2}z^{-1}\right)^{10}}{1 - \frac{1}{2}z^{-1}} \quad \text{ROC: } |z| > 0$$

(b) Solution:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^{|n|} z^{-n} = \sum_{n=-\infty}^{-1} \left(\frac{1}{2}\right)^{-n} z^{-n} + \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^{-n}$$
$$= \sum_{n=1}^{\infty} \left(\frac{z}{2}\right)^n + \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^{-n} = \frac{\frac{z}{2}}{1 - \frac{1}{2}z} + \frac{1}{1 - \frac{1}{2}z^{-1}}$$
$$= \frac{-1}{1 - 2z^{-1}} + \frac{1}{1 - \frac{1}{2}z^{-1}} = \frac{-\frac{3}{2}z^{-1}}{1 - \frac{5}{2}z^{-1} + z^{-2}} \quad \text{ROC: } \frac{1}{2} < |z| < 2$$

(c) Solution:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=-\infty}^{\infty} 5^{|n|}z^{-n} = \sum_{n=-\infty}^{-1} 5^{-n}z^{-n} + \sum_{n=0}^{\infty} 5^n z^{-n}$$
$$= \frac{5z}{1 - 5z} + \frac{1}{1 - 5z^{-1}} = \frac{\frac{24}{5}z^{-1}}{1 - \frac{26}{5}z^{-1} + z^{-2}} \quad \text{ROC: } |z| \in \phi$$

(d) Solution:

$$x[n] = \left(\frac{1}{2}\right)^n \cos(\pi n/3)u[n] = x[n] = \left(\frac{1}{2}\right)^n (\frac{1}{2}e^{j\pi n/3} + \frac{1}{2}e^{-j\pi n/3})u[n]$$

$$\begin{aligned}
X(z) &= \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1}{2}e^{j\pi/3}z^{-1}\right)^n + \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1}{2}e^{-j\pi/3}z^{-1}\right)^n \\
&= \frac{1}{2}\frac{1}{1-\frac{1}{2}e^{j\pi/3}z^{-1}} + \frac{1}{2}\frac{1}{1-\frac{1}{2}e^{-j\pi/3}z^{-1}} \\
&= \frac{1-\frac{1}{2}\cos\left(\frac{\pi}{3}\right)z^{-1}}{1-\cos\left(\frac{\pi}{3}\right)z^{-1}+\frac{1}{4}z^{-2}} = \frac{1-\frac{1}{4}z^{-1}}{1-\frac{1}{2}z^{-1}+\frac{1}{4}z^{-2}} \quad \text{ROC: } |z| > \frac{1}{2}
\end{aligned}$$

2. (a) Proof:
The input in `x=filter(b,a,[1,zeros(1,N)])` is actually impulse signal $\delta[n]$. The z-transform of impulse response is 1. Hence, $Y(z) = X(z)$.

(b) Solution:

$$\begin{aligned}
X(z) &= \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=-\infty}^{\infty}\left[\left(\frac{1}{2}\right)^n + \left(-\frac{1}{3}\right)^n\right]u[n]z^{-n} \\
&= \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n z^{-n} + \sum_{n=0}^{\infty}\left(-\frac{1}{3}\right)^n z^{-n} \\
&= \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1+\frac{1}{3}z^{-1}} = \frac{2-\frac{1}{6}z^{-1}}{1-\frac{1}{6}z^{-1}-\frac{1}{6}z^{-2}} \quad \text{ROC: } |z| > \frac{1}{2}
\end{aligned}$$

(c) MATLAB script:

```
% P0302: verify z-transform expression of a causal
%        sequence using function 'filter'
close all; clc
n = 0:20;
xn = (1/2).^n+(-1/3).^n;
b = [2 -1/6];
a = [1 -1/6 -1/6];
xnz = filter(b,a,[1,zeros(1,length(n)-1)]);
% Plot
hf = figconfg('P0302');
```

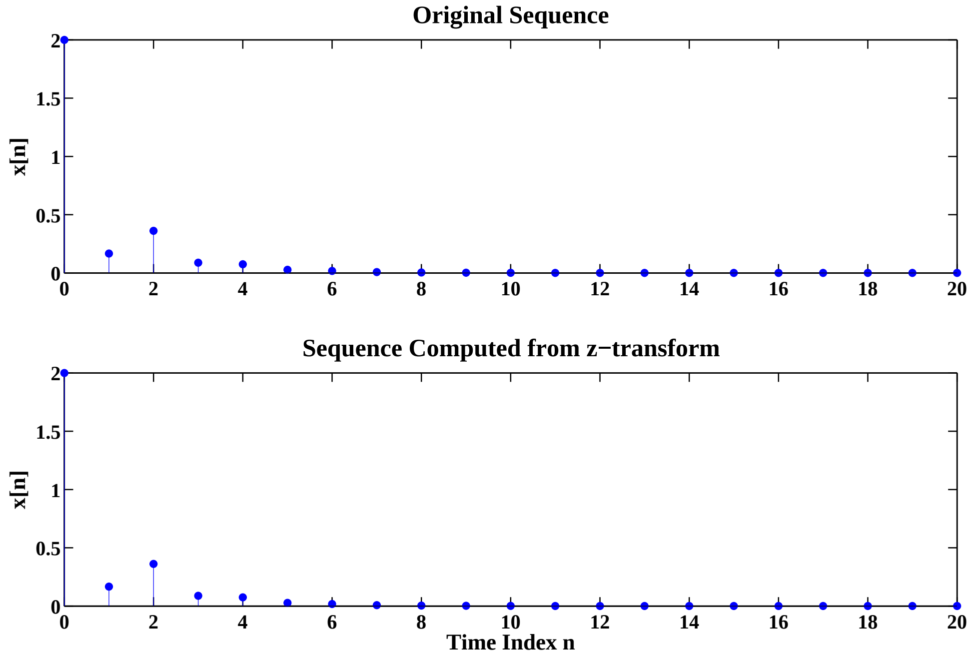

FIGURE 3.2: MATLAB verification of z-transform expression using“x=filter(b,a,[1,zeros(1,N)])”.

3. Proof:

$$x[n] = (r^n \sin \omega_0 n)u[n] = r^n \left(\frac{1}{2j} e^{j\omega_0 n} - \frac{1}{2j} e^{-j\omega_0 n} \right) u[n]$$

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \frac{1}{2j} \sum_{n=0}^{\infty} \left(r e^{j\omega_0} z^{-1} \right)^n - \frac{1}{2j} \sum_{n=0}^{\infty} \left(r e^{-j\omega_0} z^{-1} \right)^n \\ &= \frac{1}{2j} \frac{1}{1 - r e^{j\omega_0} z^{-1}} - \frac{1}{2j} \frac{1}{1 - r e^{-j\omega_0} z^{-1}} \\ &= \frac{r(\sin \omega_0) z^{-1}}{1 - 2(r \cos \omega_0) z^{-1} + r^2 z^{-2}} \quad \text{ROC: } |z| > r \end{aligned}$$

4. (a) Solution:

$$X(z) = \frac{1 - \frac{1}{3} z^{-1}}{(1 - z^{-1})(1 + 2z^{-1})} = \frac{\frac{2}{9}}{1 - z^{-1}} + \frac{\frac{7}{9}}{1 + 2z^{-1}}$$

$$\text{ROC: } |z| > 2 \quad x[n] = \frac{2}{9}u[n] + \frac{7}{9}(-2)^n u[n]$$

$$\text{ROC: } |z| < 1 \quad x[n] = -\frac{2}{9}u[-n-1] - \frac{7}{9}(-2)^n u[-n-1]$$

$$\text{ROC: } 1 < |z| < 2 \quad x[n] = \frac{2}{9}u[n] - \frac{7}{9}(-2)^n u[-n-1]$$

(b) Solution:

$$X(z) = \frac{1 - z^{-1}}{1 - \frac{1}{4}z^{-1}} = 4 - \frac{3}{1 - \frac{1}{4}z^{-1}} \quad \text{ROC: } |z| > \frac{1}{4}$$

$$x[n] = 4\delta[n] - 3\left(\frac{1}{4}\right)^n u[n]$$

(c) Solution:

$$X(z) = \frac{2}{1 - 0.5z^{-1}} + \frac{-1}{1 - 0.25z^{-1}}, \quad \text{ROC: } |z| > 0.5$$

$$x[n] = 2(0.5)^n u[n] - (0.25)^n u[n]$$

5. Solution:

$$X(z) = z^2(1 - \frac{1}{3}z^{-1})(1 - z^{-1})(1 + 2z^{-2}) = z^2 - \frac{4}{3}z + \frac{7}{3} - \frac{8}{3}z^{-1} + \frac{2}{3}z^{-2}$$

$$x[n] = \delta[n+2] - \frac{4}{3}\delta[n+1] + \frac{7}{3}\delta[n] - \frac{8}{3}\delta[n-1] + \frac{2}{3}\delta[n-2]$$

6. (a) Solution:

$$\text{Time-shifting:} \quad Y(z) = z^{-3}X(z) = \frac{z^{-3}}{1 - 2z^{-1}}, \quad \text{ROC: } |z| < 2$$

(b) Solution:

$$\text{Scaling:} \quad Y(z) = X(3z) = \frac{1}{1 - \frac{2}{3}z^{-1}}, \quad \text{ROC: } |z| < \frac{2}{3}$$

(c) Solution:

$$\text{Folding and convolution:} \quad Y(z) = X(z)X(1/z) = \frac{1}{1 - 2z^{-1}} \cdot \frac{1}{1 - 2z}$$

$$= \frac{-\frac{1}{2}}{1 - \frac{5}{2}z^{-1} + z^{-2}}, \quad \text{ROC: } \frac{1}{2} < |z| < 2$$

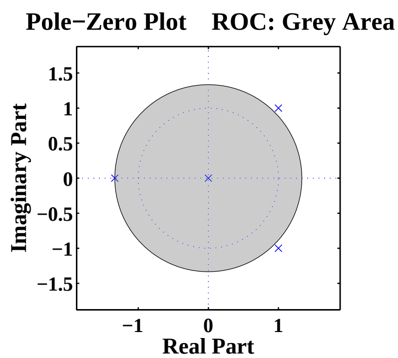

FIGURE 3.3: Pole-zero plot and ROC of $y[n]$.

11. (a) Solution:

$$H(z) = \frac{1}{1-az^{-1}}, \quad \text{ROC: } |z| > |a|$$

$$X(z) = \frac{1}{1-bz^{-1}}, \quad \text{ROC: } |z| > |b|$$

$$\begin{aligned} Y(z) = H(z)X(z) &= \frac{1}{1-az^{-1}} \cdot \frac{1}{1-bz^{-1}} \\ &= \frac{\frac{a}{a-b}}{1-az^{-1}} + \frac{\frac{-b}{a-b}}{1-bz^{-1}}, \quad \text{ROC: } |z| > \max\{|a|, |b|\} \end{aligned}$$

$$y[n] = \frac{a}{a-b} \cdot a^n u[n] + \frac{b}{b-a} \cdot b^n u[n]$$

(b) Solution:

$$H(z) = X(z) = \frac{1}{1-az^{-1}}, \quad \text{ROC: } |z| > |a|$$

$$\begin{aligned} Y(z) = H(z)X(z) &= \frac{1}{(1-az^{-1})^2} \\ &= \frac{1}{1-az^{-1}} + \frac{az^{-1}}{1-az^{-1}}, \quad \text{ROC: } |z| > |a| \end{aligned}$$

$$y[n] = a^n u[n] + na^n u[n]$$

(c) Solution:

$$x[n] = a^{-n}u[-n] = \delta[n] + (a^{-1})^n u[-n-1]$$

$$X(z) = 1 + \frac{-1}{1-a^{-1}z^{-1}}, \quad \text{ROC: } |z| < |a|^{-1}$$

$$\begin{aligned} Y(z) = H(z)X(z) &= \left(\frac{1}{1-az^{-1}}\right) \cdot \left(1 + \frac{-1}{1-a^{-1}z^{-1}}\right) \\ &= \frac{\frac{1}{1-a^2}}{1-az^{-1}} + \frac{\frac{-1}{1-a^2}}{1-a^{-1}z^{-1}}, \quad \text{ROC: } |a| < |z| < |a|^{-1} \end{aligned}$$

$$y[n] = \frac{a^n}{1-a^2}u[n] + \frac{a^{-n}}{1-a^2}u[-n-1] = \frac{a^{|n|}}{1-a^2}$$

12. (a) Proof:

$$r_{xx}[\ell] \triangleq \sum_n x[n]x[n-\ell] = \sum_n x[n+\ell]x[n] = x[l] * x[-l]$$

By applying the folding property, the z-transform of sequence $x[-l]$ is $X(z^{-1})$ with ROC $\beta^{-1} < |z| < \alpha^{-1}$. Hence, we proved

$$R_{xx}(z) = X(z)X\left(z^{-1}\right) \quad \text{ROC: } \max\{\alpha, \beta^{-1}\} < |z| < \min\{\beta, \alpha^{-1}\}$$

(b) Solution:

$$X(z) = \frac{1}{1-az^{-1}}, \quad \text{ROC: } |z| > |a|$$

$$X(z^{-1}) = \frac{1}{1-az} = \frac{-az^{-1}}{1-a^{-1}z^{-1}}, \quad \text{ROC: } |z| < |a|^{-1}$$

$$\begin{aligned} R_{xx}(z) = X(z)X\left(z^{-1}\right) &= \left(\frac{1}{1-az^{-1}}\right)\left(\frac{-az^{-1}}{1-a^{-1}z^{-1}}\right) \\ &= \frac{-az^{-1}}{1-(a+a^{-1})z^{-1}+z^{-2}}, \quad \text{ROC: } |a| < |z| < |a|^{-1} \end{aligned}$$

(c) Solution:

$$R_{xx}(z) = \frac{\frac{a^2}{1-a^2}}{1-az^{-1}} + \frac{\frac{-a^2}{1-a^2}}{1-a^{-1}z^{-1}}, \quad \text{ROC: } |a| < |z| < |a|^{-1}$$

$$r_{xx}[\ell] = \left(\frac{a^2}{1-a^2}\right)a^l u[l] + \left(\frac{a^2}{1-a^2}\right)a^{-l}u[-l-1] = \left(\frac{a^2}{1-a^2}\right)a^{|l|}$$

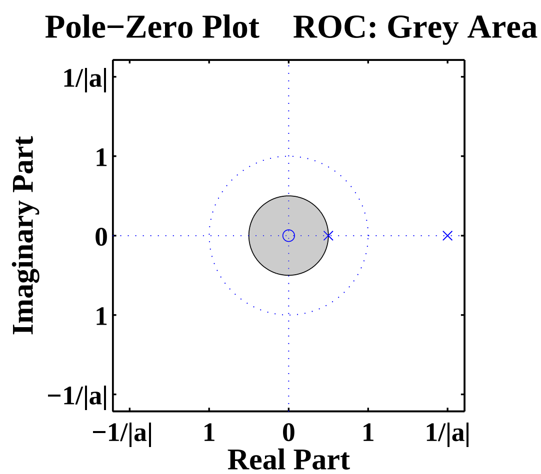

FIGURE 3.4: Pole-zero plot and ROC of $R_{xx}(z)$.

13. Solution:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-1}}{1-\frac{5}{2}z^{-1}+z^{-2}} = \frac{\frac{2}{3}}{1-2z^{-1}} + \frac{-\frac{2}{3}}{1-\frac{1}{2}z^{-1}}$$

ROC: $|z| > 2$

$$h[n] = \frac{2}{3}\cdot 2^n u[n] - \frac{2}{3}\cdot\left(\frac{1}{2}\right)^n u[n]$$

ROC: $|z| < \frac{1}{2}$

$$h[n] = -\frac{2}{3}\cdot 2^n u[-n-1] + \frac{2}{3}\cdot\left(\frac{1}{2}\right)^n u[-n-1]$$

ROC: $\frac{1}{2} < |z| < 2$

$$h[n] = -\frac{2}{3}\cdot 2^n u[-n-1] - \frac{2}{3}\cdot\left(\frac{1}{2}\right)^n u[n]$$

14. Solution:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1-\frac{1}{2}z^{-1}}, \quad \text{ROC: } |z| > \frac{1}{2}$$

(a) $x[n] = \mathrm{e}^{\mathrm{j}(\pi/4)n}, \quad -\infty < n < \infty$

$$y[n] = \mathrm{e}^{\mathrm{j}(\pi/4)n} \cdot H(z)\,|_{z=\mathrm{e}^{\mathrm{j}(\pi/4)}} = \frac{1}{1-\frac{1}{2}e^{-\mathrm{j}(\pi/4)}} e^{\mathrm{j}(\pi/4)n}$$

(b) $x[n] = \mathrm{e}^{\mathrm{j}(\pi/4)n} u[n]$

$$X(z) = \frac{1}{1-\mathrm{e}^{\mathrm{j}(\pi/4)}z^{-1}}, \quad \text{ROC: } |z| > 1$$

$$Y(z) = X(z)H(z) = \left(\frac{1}{1-\frac{1}{2}z^{-1}}\right)\left(\frac{1}{1-\mathrm{e}^{\mathrm{j}(\pi/4)}z^{-1}}\right)$$
$$= \frac{\frac{1}{1-2\mathrm{e}^{\mathrm{j}(\pi/4)}}}{1-\frac{1}{2}z^{-1}} + \frac{\frac{1}{1-2\mathrm{e}^{-\mathrm{j}(\pi/4)}}}{1-\mathrm{e}^{\mathrm{j}(\pi/4)}z^{-1}}, \quad \text{ROC: } |z| > 1$$

$$y[n] = \frac{1}{1-2\mathrm{e}^{\mathrm{j}(\pi/4)}}\left(\frac{1}{2}\right)^n u[n] + \frac{1}{1-2\mathrm{e}^{-\mathrm{j}(\pi/4)}}\mathrm{e}^{\mathrm{j}(\pi/4)n}u[n]$$

(c) $x[n] = (-1)^n, \quad -\infty < n < \infty$

$$y[n] = (-1)^n \cdot H(z)\,|_{z=-1} = \frac{2}{3}(-1)^n$$

(d) $x[n] = (-1)^n u[n]$

$$X(z) = \frac{1}{1+z^{-1}}, \quad \text{ROC: } |z| > 1$$

$$Y(z) = X(z)H(z) = \left(\frac{1}{1-\frac{1}{2}z^{-1}}\right)\left(\frac{1}{1+z^{-1}}\right)$$
$$= \frac{\frac{1}{3}}{1-\frac{1}{2}z^{-1}} + \frac{\frac{2}{3}}{1+z^{-1}}, \quad \text{ROC: } |z| > 1$$
$$y[n] = \frac{1}{3}\left(\frac{1}{2}\right)^n u[n] + \frac{2}{3}(-1)^n u[n]$$

15. (a) Solution:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1-\frac{3}{4}z^{-1}-\frac{1}{8}z^{-1}}$$
$$= \frac{1}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)}, \quad \text{ROC: } |z| > \frac{1}{2}$$

The system is stable if it is causal.

(b) Solution:

$$H(z) = \frac{-1}{1-\frac{1}{4}z^{-1}} + \frac{2}{1-\frac{1}{2}z^{-1}}, \quad \text{ROC: } |z| > \frac{1}{2}$$

$$h[n] = -\left(\frac{1}{4}\right)^n u[n] + 2\left(\frac{1}{2}\right)^n u[n]$$

(c) Solution:

$$U(z) = \frac{1}{1-z^{-1}}, \quad \text{ROC: } |z| > 1$$

$$\begin{aligned} S(z) = U(z)X(z) &= \frac{1}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)\left(1-z^{-1}\right)}, \\ &= \frac{\frac{1}{3}}{1-\frac{1}{4}z^{-1}} + \frac{-2}{1-\frac{1}{2}z^{-1}} + \frac{\frac{8}{3}}{1-z^{-1}}, \quad \text{ROC: } |z| > \frac{1}{2} \end{aligned}$$

$$s[n] = \frac{1}{3}\left(\frac{1}{4}\right)^n u[n] - 2\left(\frac{1}{2}\right)^n u[n] + \frac{8}{3}u[n]$$

(d) MATLAB script:

```
% P0315: verify the calculated impulse and step
%        response sequences using function 'filter'
close all; clc
n = 0:10;
%% Impluse Response:
hn = -(1/4).^n + 2*(1/2).^n ;
b = 1;
a = [1 -3/4 1/8];
hnz = filter(b,a,[1,zeros(1,length(n)-1)]);
% Plot
hf1 = figconfg('P0315a');
subplot(211)
stem(n,hn,'filled')
ylabel('h[n]','fontsize',LFS)
title('Impulse Response Expression Sequence','fontsize',TFS)
subplot(212)
stem(n,hnz,'filled')
xlabel('Time Index n','fontsize',LFS)
ylabel('h[n]','fontsize',LFS)
title('Sequence Computed from z-transform','fontsize',TFS)
```

```
%% Step Response:
un = 1/3*(1/4).^n - 2*(1/2).^n +8/3;
b = 1;
a = [1 -7/4 7/8 -1/8];
unz = filter(b,a,[1,zeros(1,length(n)-1)]);
% Plot
hf2 = figconfg('P0315b');
subplot(211)
stem(n,un,'filled')
ylabel('s[n]','fontsize',LFS)
title('Step Response Expression Sequence','fontsize',TFS)
subplot(212)
stem(n,unz,'filled')
xlabel('Time Index n','fontsize',LFS)
ylabel('s[n]','fontsize',LFS)
title('Sequence Computed from z-transform','fontsize',TFS)
```

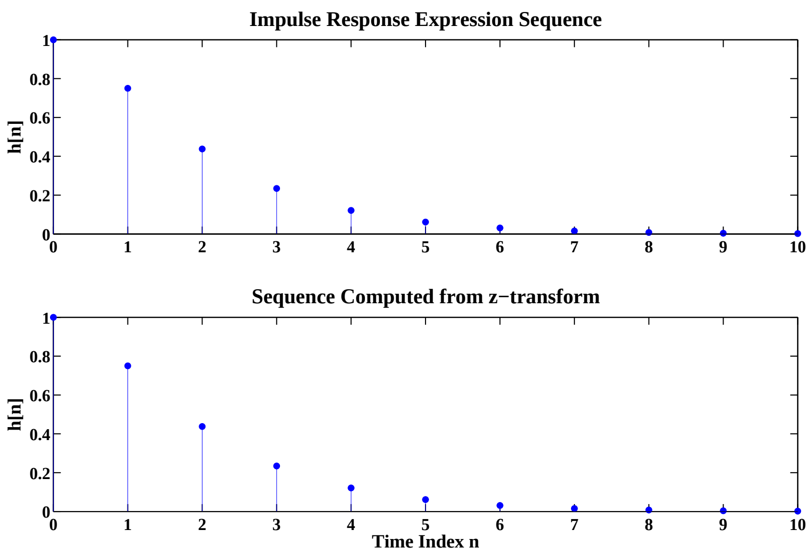

FIGURE 3.5: MATLAB verification of the impulse response expression obtained in part (b).

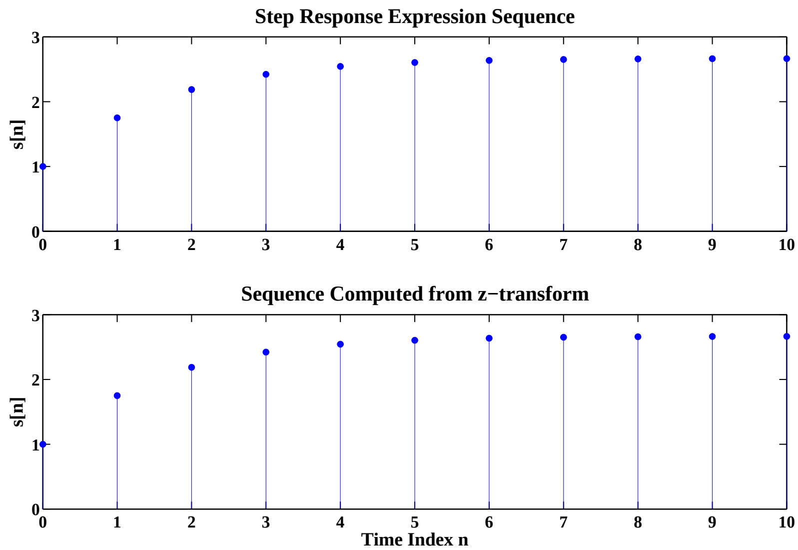

FIGURE 3.6: MATLAB verification of the step response expression obtained in part (c).

16. (a) Solution:

$$X(z) = \frac{1}{1 - z^{-1}}, \quad \text{ROC: } |z| > 1$$

$$Y(z) = \frac{2}{1 - \frac{1}{3}z^{-1}}, \quad \text{ROC: } |z| > \frac{1}{3}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{2(1 - z^{-1})}{1 - \frac{1}{3}z^{-1}} = 6 + \frac{-4}{1 - \frac{1}{3}z^{-1}}, \quad \text{ROC: } |z| > \frac{1}{3}$$

$$h[n] = 6\delta[n] - 4\left(\frac{1}{3}\right)^n u[n]$$

(b) Solution:

$$X(z) = \frac{1}{1 - \frac{1}{2}z^{-1}}, \quad \text{ROC: } |z| > \frac{1}{2}$$

$$Y(z) = X(z)H(z) = \frac{2(1 - z^{-1})}{\left(1 - \frac{1}{3}z^{-1}\right)\left(1 - \frac{1}{2}z^{-1}\right)}$$

$$= \frac{8}{1 - \frac{1}{3}z^{-1}} + \frac{-6}{1 - \frac{1}{2}z^{-1}}, \quad \text{ROC: } |z| > \frac{1}{2}$$

19. (a) Solution:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - \frac{1}{1024}z^{-10}}{1 - \frac{1}{2}z^{-1}}$$

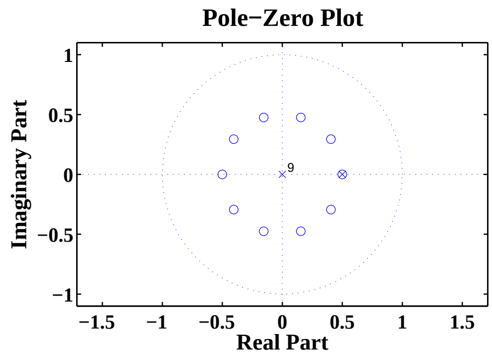

FIGURE 3.10: Pole-zero pattern of the system.

(b) Impulse response.

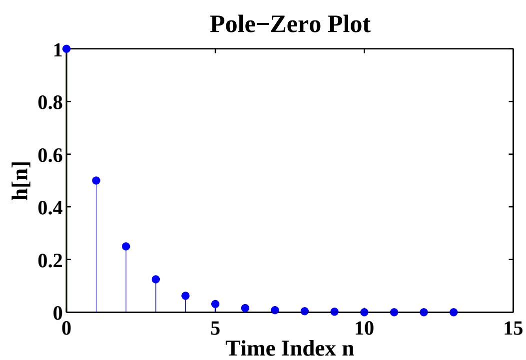

FIGURE 3.11: Impulse response $h[n]$.

(c) Comments: The pole at $z = \frac{1}{2}$ is canceled and the ROC is all the z plane. The corresponding time sequence should be of finite length.

(d) Solution:

$$H(z) = \sum_{m=0}^{9} 2^{-m} x[n-m]$$

If ROC: $|z| > 3.7443$

$$x[n] = \delta[n] + 0.5383(3.7443)^n u[n] + 2 \times 6.6714 \cos(0.9041n + 1.0437)(0.2067)^n u[n]$$

If ROC: $|z| < 0.2067$

$$x[n] = \delta[n] - 0.5383(3.7443)^n u[-n-1] - 2 \times 6.6714 \cos(0.9041n + 1.0437)(0.2067)^n u[-n-1]$$

If ROC: $0.2067 < |z| < 3.7443$

$$x[n] = \delta[n] - 0.5383(3.7443)^n u[-n-1] + 2 \times 6.6714 \cos(0.9041n + 1.0437)(0.2067)^n u[n]$$

(b) Solution:

$$X(z) = \frac{4}{1+z^{-1}} + \frac{-4}{1+\frac{1}{2}z^{-1}} - 4\frac{-\frac{1}{2}z^{-1}}{\left(1+\frac{1}{2}z^{-1}\right)^2}$$

$$x[n] = 4(-1)^n u[n] - 4\left(-\frac{1}{2}\right)^n u[n] - 4n\left(-\frac{1}{2}\right)^n u[n]$$

(c) tba

29. (a) Solution:

$$X(z) = 2\frac{1-\frac{1}{2}z^{-1}}{1-z^{-1}+0.81z^{-2}} + \frac{20}{\sqrt{14}}\frac{\frac{\sqrt{14}}{5}z^{-1}}{1-z^{-1}+0.81z^{-2}}$$

$$x[n] = 2(0.9)^n \cos(\omega_0 n)u[n] + \frac{20}{\sqrt{14}}(0.9)^n \sin(\omega_0 n)u[n], \ \cos\omega_0 = \frac{5}{9}$$

(b) See plot below.

30. (a) Solution:

$$\text{Time shifting:} \quad Y(z) = z^{-2}X(z) = \frac{z^{-2}}{1-\frac{1}{3}z^{-1}} \quad \text{ROC: } |z| > \frac{1}{3}$$

(b) Solution:

$$\text{Scaling:} \quad Y(z) = X(z/2) = \frac{1}{1-\frac{2}{3}z^{-1}} \quad \text{ROC: } |z| > \frac{2}{3}$$

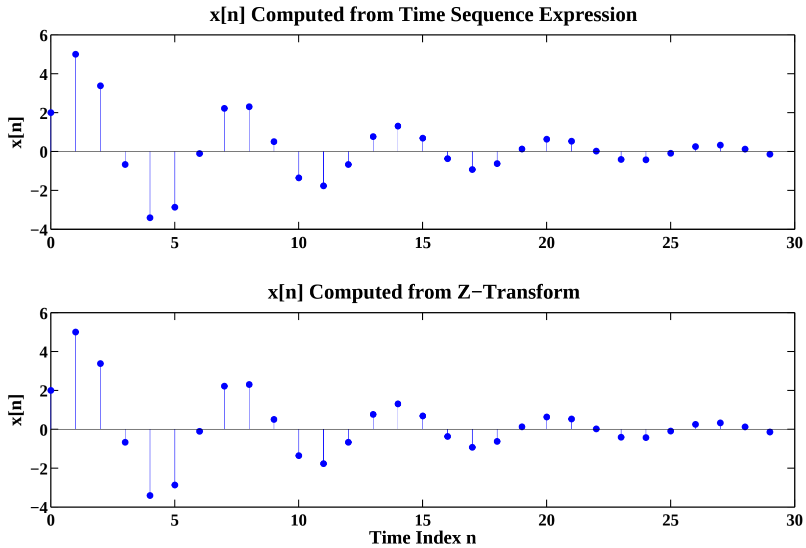

FIGURE 3.24: MATLAB verification of $x[n]$ expression obtained in part (a).

(c) Solution:

Convolution, time shifting and folding:

$$Y(z) = X(z)X(1/z)z = \frac{-3}{1-3z^{-1}}\frac{1}{1-\frac{1}{3}z^{-1}} = \frac{-3}{1-\frac{10}{3}+z^{-2}}$$

ROC: $\frac{1}{3} < |z| < \frac{2}{3}$

(d) Solution:

$$\text{Differentiation:}\quad Y(z) = \frac{\frac{1}{3}z\left(z+\frac{1}{3}\right)}{z-\frac{1}{3}} \quad \text{ROC: } |z| > \frac{1}{3}$$

(e) Solution:

Linearity and time shifting:

$$Y(z) = \frac{2z}{1-\frac{1}{3}z^{-1}} + \frac{3z^{-3}}{1-\frac{1}{3}z^{-1}} = \frac{2z+3z^{-3}}{1-\frac{1}{3}z^{-1}} \quad \text{ROC: } |z| > \frac{1}{3}$$

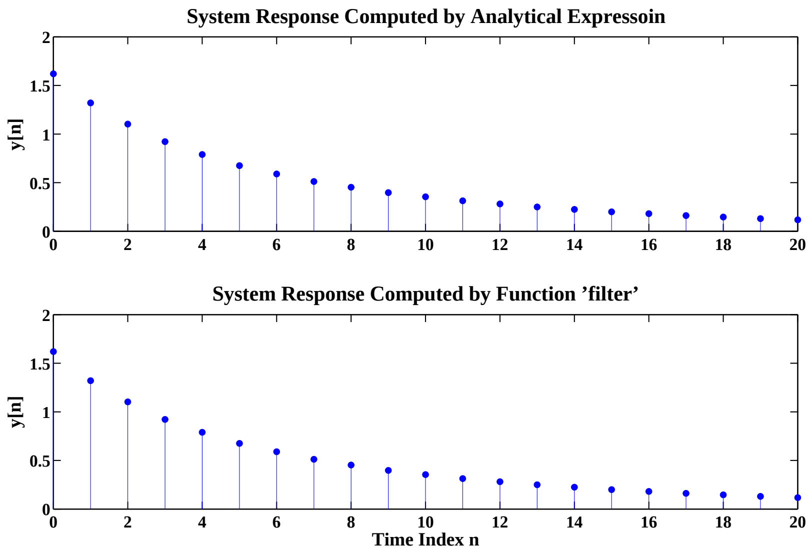

FIGURE 3.29: MATLAB verification of system response $y[n]$.

44. Solution:

$$Y^+(z) = \frac{1}{4}(y[-1] + z^{-1}Y^+(z)) + X^+(z) + 3(x[-1] + z^{-1}X^+(z))$$

$$Y^+(z) = \frac{1}{4}(2 + z^{-1}Y^+(z)) + X^+(z) + 3(0 + z^{-1}X^+(z))$$

$$Y^+(z) = \frac{\frac{1}{2}}{1 - \frac{1}{4}z^{-1}} + \frac{1 + 3z^{-1}}{1 - \frac{1}{4}z^{-1}}X^+(z)$$

$$X^+(z) = \frac{1}{1 - \mathrm{e}^{\mathrm{j}\pi/4}z^{-1}}, \quad \text{ROC: } |z| > 1$$

$$y[n] = \frac{1}{2}\cdot\left(\frac{1}{4}\right).^n u[n] + (3.0955 - 3.2416\mathrm{j})\mathrm{e}^{\mathrm{j}\pi n/4}u[n] + (-2.0955 + 3.2416\mathrm{j})\left(\frac{1}{4}\right).^n u[n]$$

zero-input response:

$$\frac{1}{2}\cdot\left(\frac{1}{4}\right).^n u[n]$$

zero-state response

$$(3.0955 - 3.2416\mathrm{j})\mathrm{e}^{\mathrm{j}\pi n/4}u[n] + (-2.0955 + 3.2416\mathrm{j})\left(\frac{1}{4}\right).^n u[n]$$

transient response:

$$\frac{1}{2} \cdot \left(\frac{1}{4}\right).^{n} u[n] + (-2.0955 + 3.2416\mathrm{j}) \left(\frac{1}{4}\right).^{n} u[n]$$

steady-state response:

$$(3.0955 - 3.2416\mathrm{j})\mathrm{e}^{\mathrm{j}\pi n/4} u[n]$$

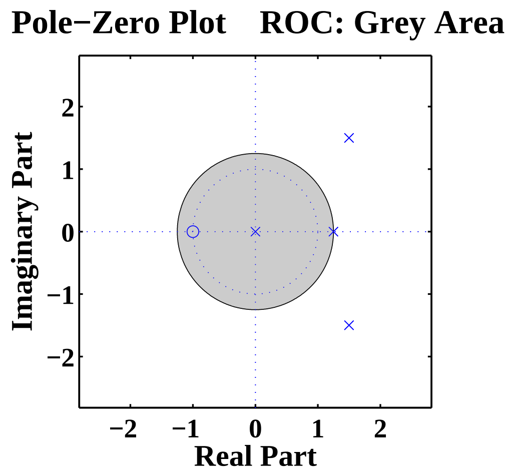

FIGURE 3.32: Pole-zero plot and ROC of $y[n] = x[-n+2]$.

53. (a) Solution:

$$H(z) = \frac{1}{1-\frac{1}{2}z^{-1}}, \quad \text{ROC: } |z| > \frac{1}{2}$$

$$X(z) = 1 + \frac{-1}{1-2z^{-1}}, \quad \text{ROC: } |z| < 2$$

$$Y(z) = H(z)X(z) = \frac{-4/5}{1-2z^{-1}} + \frac{4/5}{1-\frac{1}{2}z^{-1}}, \quad \text{ROC: } \frac{1}{2} < |z| < 2$$

$$y[n] = \left(\frac{4}{5}\right) n2^n u[-n-1] + \left(\frac{4}{5}\right)\left(\frac{1}{2}\right)^n u[n]$$

(b) Solution:

$$H(z) = \frac{1}{1-\frac{1}{3}z^{-1}} + \frac{-1}{1-3z^{-1}}, \quad \text{ROC: } \frac{1}{3} < |z| < 3$$

$$X(z) = \frac{1}{1-\frac{1}{2}z^{-1}}, \quad \text{ROC: } |z| > \frac{1}{2}$$

$$Y(z) = H(z)X(z) = \frac{3/20}{1-3z^{-1}} + \frac{13/5}{1-\frac{1}{2}z^{-1}} + \frac{-7/4}{1-\frac{1}{3}z^{-1}}, \quad \text{ROC: } \frac{1}{2} < |z| < 3$$

$$y[n] = -\left(\frac{3}{20}\right) \cdot 3^n u[-n-1] + \left(\frac{13}{5}\right)\left(\frac{1}{2}\right)^n u[n] - \left(\frac{7}{4}\right)\left(\frac{1}{3}\right)^n u[n]$$

(c) Solution:

$$H(z) = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1-2z^{-1}}, \quad \text{ROC: } \frac{1}{2} < |z| < 2$$

$$X(z) = \frac{1}{1-\frac{1}{4}z^{-1}} + \frac{1}{1-4z^{-1}}, \quad \text{ROC: } \frac{1}{4} < |z| < 4$$

$$\begin{aligned} Y(z) = H(z)X(z) &= \frac{-2/35}{1-4z^{-1}} + \frac{-3/7}{1-2z^{-1}} + \frac{20/7}{1-\frac{1}{2}z^{-1}} \\ &\quad + \frac{-48/35}{1-\frac{1}{4}z^{-1}}, \quad \text{ROC: } \frac{1}{2} < |z| < 2 \end{aligned}$$

$$\begin{aligned} y[n] &= \left(\frac{2}{35}\right) \cdot 4^n u[-n-1] + \left(\frac{3}{7}\right) \cdot 2^n u[-n-1] \\ &\quad + \left(\frac{20}{7}\right)\left(\frac{1}{2}\right)^n u[n] + \left(-\frac{48}{35}\right)\left(\frac{1}{4}\right)^n u[n] \end{aligned}$$

54. (a) Solution:

$$X(z) = \frac{-1}{1-bz^{-1}}, \quad \text{ROC: } |z| < |b|$$

$$X(z^{-1}) = \frac{-1}{1-bz}, \quad \text{ROC: } |z| > |b|^{-1}$$

$$R_{xx}(z) = X(z)X(1/z) = \frac{-b^{-1}z^{-1}}{(1-bz^{-1})(1-b^{-1}z^{-1})}, \quad \text{ROC: } |b|^{-1} < |z| < |b|$$

(b) See plot below.

(c) Solution:

$$R_{xx}(z) = \frac{\frac{-b^{-2}}{1-b^{-2}}}{1-bz^{-1}} + \frac{\frac{b^{-2}}{1-b^{-2}}}{1-b^{-1}z^{-1}}, \quad \text{ROC: } |b|^{-1} < |z| < |b|$$

$$\begin{aligned} r_{xx}[\ell] &= \left(\frac{b^{-2}}{1-b^{-2}}\right) b^{\ell} u[-\ell-1] + \left(\frac{b^{-2}}{1-b^{-2}}\right) b^{-\ell} u[\ell] \\ &= \left(\frac{b^{-2}}{1-b^{-2}}\right) b^{-|\ell|} \end{aligned}$$

59. (a) Solution:

$$X(z) = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{-1}{1-2z^{-1}} = \frac{-\frac{3}{2}z^{-1}}{1-\frac{5}{2}z^{-1}+z^{-2}}, \quad \text{ROC: } \frac{1}{2} < |z| < 2$$

$$Y(z) = \frac{3}{1-0.7z^{-1}}, \quad \text{ROC: } |z| > 0.7$$

$$H(z) = \frac{Y(z)}{X(z)} = -2z + \frac{20}{7} + \frac{26/35}{1-0.7z^{-1}}, \quad \text{ROC: } |z| > 0.7$$

$$h[n] = -2\delta[n+1] + \frac{20}{7}\delta[n] + \frac{26}{35}(0.7)^n u[n]$$

(b) Solution:

$$X(z) = 1 + \frac{-1}{1-0.9z^{-1}}, \quad \text{ROC: } |z| < 0.9$$

$$y[n] = -\frac{20}{9}\delta[n+1] + \frac{20}{7}\delta[n] + \left(\frac{117}{35}\right)(0.9)^n u[-n-1] + \left(\frac{117}{35}\right)(0.7)^n u[n]$$

60. (a) Solution:

$$X(z) = \frac{1}{1-\frac{1}{3}z^{-1}} + \frac{-1}{1-3z^{-1}} = \frac{-\frac{8}{3}z^{-1}}{1-\frac{10}{3}z^{-1}+z^{-2}}, \quad \text{ROC: } \frac{1}{3} < |z| < 3$$

$$\begin{aligned} Y(z) &= \frac{2}{1-\frac{1}{3}z^{-1}} - 4 + \frac{4}{1-4z^{-1}} \\ &= \frac{2+8z^{-1}-\frac{16}{3}z^{-2}}{(1-\frac{1}{3}z^{-1})(1-4z^{-1})}, \quad \text{ROC: } \frac{1}{3} < |z| < 4 \end{aligned}$$

$$H(z) = \frac{Y(z)}{X(z)} = -\frac{3}{4}z - \frac{19}{8} + \frac{3}{2}z^{-1} + \frac{-\frac{5}{4}}{1-4z^{-1}}, \quad \text{ROC: } |z| < 4$$

$$h[n] = -\frac{3}{4}\delta[n+1] - \frac{19}{8}\delta[n] + \frac{3}{2}\delta[n-1] + \left(\frac{5}{4}\right) \cdot 4^n u[-n-1]$$

(b) Solution:

$$X(z) = \frac{1}{1-\frac{3}{4}z^{-1}}, \quad \text{ROC: } |z| > \frac{3}{4}$$

$$Y(z) = \frac{2-\frac{19}{4}z^{-1}}{(1-\frac{3}{4}z^{-1})(1-4z^{-1})}, \quad \text{ROC: } \frac{3}{4} < |z| < 4$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{19}{16} + \frac{\frac{13}{16}}{1 - 4z^{-1}}, \quad \text{ROC: } |z| < 4$$

$$h[n] = \frac{19}{16}\delta[n] + \left(-\frac{13}{16}\right) \cdot 4^n u[-n-1]$$

61. (a) Solution:

$$X(z) = \frac{-\frac{7}{6} + \frac{31}{18}z^{-1}}{1 - \frac{10}{3}z^{-1} + z^{-2}}, \quad \text{ROC: } \frac{1}{3} < |z| < 3$$

(b) Soluton:

$$H(z) = \frac{Y(z)}{X(z)} = -\frac{\frac{20}{11}}{1 - 2z^{-1}} + \frac{\frac{759}{260}}{1 - \frac{31}{21}z^{-1}} - \frac{\frac{10}{41}}{1 - \frac{1}{2}z^{-1}}$$

If ROC $|z| > 2$,

$$h[n] = -\frac{20}{11} \cdot 2^n u[n] + \frac{759}{260}\left(\frac{31}{21}\right)^n u[n] - \frac{10}{41}\left(\frac{1}{2}\right)^n u[n]$$

If ROC $\frac{31}{21} < |z| < 2$,

$$h[n] = \frac{20}{11} \cdot 2^n u[-n-1] + \frac{759}{260}\left(\frac{31}{21}\right)^n u[n] - \frac{10}{41}\left(\frac{1}{2}\right)^n u[n]$$

If ROC $\frac{1}{2} < |z| < \frac{31}{21}$,

$$h[n] = \frac{20}{11} \cdot 2^n u[-n-1] - \frac{759}{260}\left(\frac{31}{21}\right)^n u[-n-1] - \frac{10}{41}\left(\frac{1}{2}\right)^n u[n]$$

If ROC $|z| < \frac{1}{2}$,

$$h[n] = \frac{20}{11} \cdot 2^n u[-n-1] - \frac{759}{260}\left(\frac{31}{21}\right)^n u[-n-1] + \frac{10}{41}\left(\frac{1}{2}\right)^n u[-n-1]$$

62. (a) Solution:

$$H(z) = \frac{A(1 + \mathrm{j}z^{-1})(1 - \mathrm{j}z^{-1})}{(1 - \frac{-1+\mathrm{j}}{2}z^{-1})(1 - \frac{-1-\mathrm{j}}{2}z^{-1})}$$

$$= \frac{A(1 + z^{-2})}{1 - z^{-1} + \frac{1}{2}z^{-2}}, \quad \text{ROC: } |z| > \frac{\sqrt{2}}{2}$$

$$H(1) = 0.8 \implies A = \frac{1}{5}$$