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Exam 3 Redo
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Problem 1

a) Direct multiplication of two complex numbers (a+jb)(c+jd) requires four real multiplications and two real additions. By properly arranging terms, show that it is possible to obtain the above multiplication using three real multiplications and five real additions.

First take a look at the four real multiplications and two adds...

$$\begin{aligned}(a + jb)(c + jd) &= ac + jad + jbc - bd \\ &= (ac - bd) + j(ad + bc)\end{aligned}$$

Now we identify an expression that gives the final result and uses three multiplies and five adds. Therefore we define the following three variables.

$$\begin{aligned}1) \quad m &= a(d - c) \\ &= ad - ac\end{aligned}$$

$$\begin{aligned}2) \quad n &= b(c + d) \\ &= bc + bd\end{aligned}$$

$$\begin{aligned}3) \quad p &= c(a + b) \\ &= ac + bc\end{aligned}$$

Now that the variables for the equation are identified, using these variables we will replicate the answer given from the four multiplies two adds.
First to model the first half of (ac-bd), followed by (ad+bc)

$$\begin{aligned}p - n &= ac + bc - bc + bd \\ &= ac - bd\end{aligned}$$

$$\begin{aligned}m + p &= ad - ac + ac + bc \\ &= ad + bc\end{aligned}$$

Therefore, the expression that could be expressed that is three real multiplications and five real adds is:

$$\begin{aligned}(a + jb)(c + jd) &= (p - n) + (m + p) \\ &= (c(a + b) - b(c + d)) + (a(d - c) + c(a + b))\end{aligned}$$

b) Explain the key concepts behind the Fast Fourier Transform that allow a Discrete Fourier Transform to be computed faster and yet achieve the exact same result.

There are two properties that allow the DFT to compute the FFT faster and achieve the same result. The DFT is first identified as:

$$X(k) = \sum_{n=0}^{N-1} x(n)W_N^{kn}, \quad k = 0, 1, 2, ..., N-1$$
$$where \quad W_N = e^{-j\frac{2\pi}{N}} \quad and \quad W_N^{kn} = e^{-j\frac{2\pi kn}{N}}$$

1)It is important for the DFT to maintain symmetry across the imaginary axis, such as:

$$W_N^{k+\frac{N}{2}} = -W_N^k$$

2)The DFT must maintain periodicity:

$$W_N^{k+N} = W_N^k$$

This algorithm makes the DFT have to compute far less multiply adds, and therefore faster. The algorithm used, capitalizes on the use of symmetry about the unit circle to simplify computations.

Problem 2 : FIR Filter Design

The Hann window function can be written as $w[n] = [0.5 - 0.5cos(\frac{2\pi n}{M})]W_R[n]$ where $W_R[n]$ is the rectangular window of length $M + 1$.

a) Express the DTFT of $w[n]$ in terms of the DTFT of $W_R[n]$.

$$\begin{aligned} w[n] &= [0.5 - 0.5cos\left(\frac{2\pi n}{M}\right)]W_R[n] \\ &= [0.5 - 0.5(0.5(e^{j\frac{2\pi}{M}} - e^{-j\frac{2\pi}{M}})]W_R[n] \\ &= [0.5W_R[n] - 0.25W_R[n]e^{j\frac{2\pi}{M}} + 0.25W_R[n]e^{-j\frac{2\pi}{M}} \end{aligned}$$

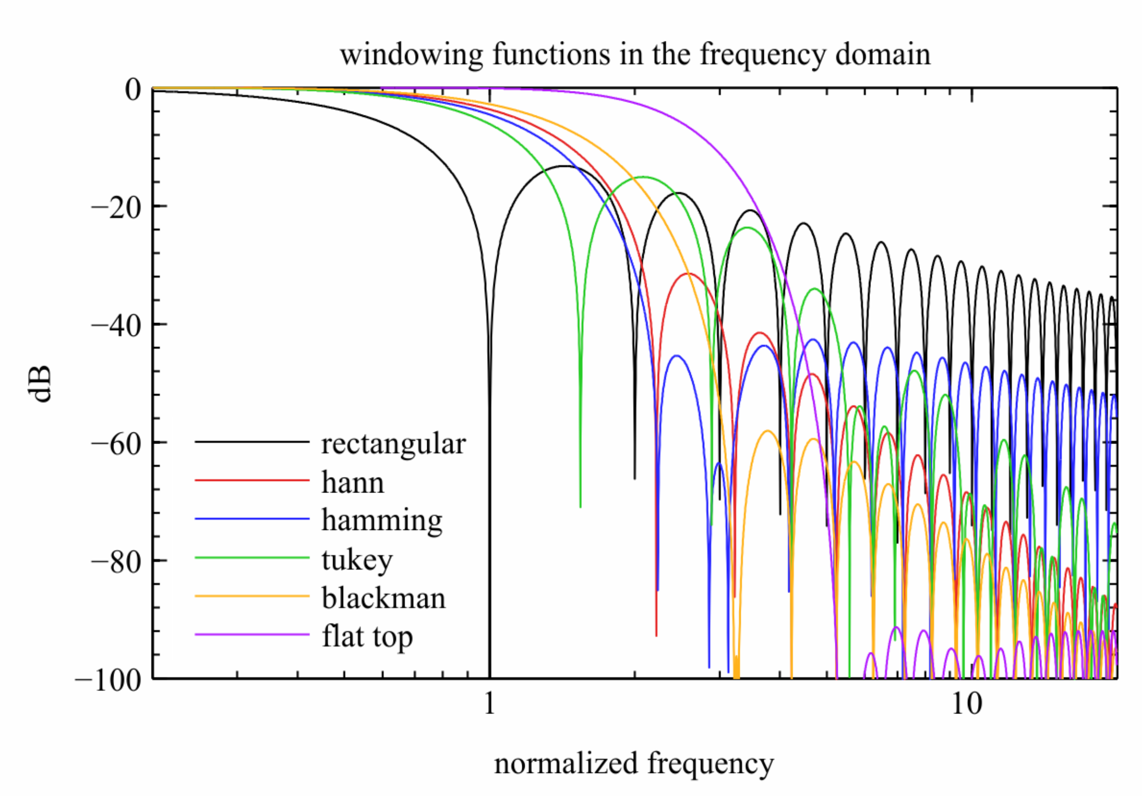
This equation could then be converted for use in the frequency domain.

$$W(e^{j\omega}) = 0.5W_R(e^{j\omega}) - 0.25W_R(e^{j(\omega-\frac{\pi}{M})}) + 0.25W_R(e^{j(\omega+\frac{\pi}{M})})$$

b)Explain why the Hann window has the wider main lobe but lower side lobes than the rectangular window of the same length.

The first term $0.5W_R[n]$, effects the main lobe making the response wider. While the other two terms scale the side lobes, giving them a smoother roll off than that of the rectangular window.
c) Explain why the width of the main lobe is important. Give an example.

The width of the main lobe is important because it plays a crucial role when the signal is convoluted in the frequency domain. The thinner that this main lobe is, the better the approximation of the signals spectrum. The main lobe and first side lobe are used to produce the smallest distortion in the spectrum of the windowed signal. (Discrete Random Signal Processing and Filtering Primer with MATLAB) An example of having this narrow window would be the comparison of a window such as a blackman window with a rectangular window, and comparing their $d\beta$ response as shown below,from Wikipedia. Notice that the rectangular has a much broader, non sharp response, meaning more spectral leakage. The blackman window has a sharp roll off as well as a good frequency response, giving the response less spectral leakage. Though in the end it all depends on what application the window is being used for.



Problem 3

Let $x[n]$ be an input signal and $h[n]$ denote the causal and stable IIR system. First, $x[n]$ filtered through $h[n]$ to obtain output $y_1[n]$. Next, the flipped signal, $x[-n]$, is filtered through $h[n]$ to obtain $y_2[n]$. Finally, the output is computed by summing $y[n] = y_1[n]$ and $y_2[n]$.

a) Let $h_s[n]$ be the impulse response of the overall system (e.g., $x[n]$ is the input, $y[n]$ is the output). Derive an expression for $h_s[n]$ in terms of $h[n]$.

By following the description given the equation can be derived as:

$$\begin{aligned} y[n] &= x[n] * h[n] + x[-n] * h[n] \\ &= x[n] * h_s[n] \end{aligned}$$

Although the impulse response cannot be found with the $x[n]$ signal flipped. So the flipped signal and properties of IIR filters make it possible to rearrange the equation.

$$\begin{aligned} x[-n] * h[n] &= \sum_{k=-\infty}^{\infty} x[-k]h[n-k] \\ &= \sum_{k=-\infty}^{\infty} x[k]h[n+k] \\ &= x[n] * h[-n] \end{aligned}$$

This means the original equation can be written as:

$$\begin{aligned} y[n] &= x[n] * h[n] + x[n] * h[-n] \\ &= x[n](h[n] + h[-n]) \end{aligned}$$

Which can then be written as:

$$h_s[n] = h[n] + h[-n]$$

b) Determine the frequency response $h_s[n]$ and discuss any important properties you observe.

From the previous derived equation, subbing in $e^{j\omega}$ for n:

$$H_s(e^{j\omega}) = H(e^{j\omega}) + H(e^{-j\omega})$$

Since the terms are complex conjugates, it can be rewritten as:

$$\boxed{2Re[He^{j\omega})]}$$

c) Will these properties hold if $h[n]$ is an IIR filter? Explain the significance of your findings.

The properties could not hold for an IIR filter. This is because the flipped signal causes the system to lose it causality. However, the filter could still be used as this is a DSP algorithm called Forward-Backward Filtering. This process squares the amplitude response, and zeros the phase response. By sacrificing the phase response this could improve the response of the stop band.