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ECE 4522: Digital Signal Processing
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Exam No. 3

Problem No. 1: Fast Fourier

(1a) Direct multiplication of two complex numbers $(a+jb)(c+jd)$ requires four real multiplications and two real additions. By properly arranging terms, show that it is possible to obtain the above multiplication using three real multiplications and five real additions.

$$(a+jb)(c+jd) = ac + jad + jbc - bd = (ac - bd) + j(ad + bc)$$

The equations below require three multiplies and three adds.

$$k_1 = a(d-c) = ad - ac$$
$$k_2 = b(c+d) = bc + bd$$
$$k_3 = c(a+b) = ac + bc$$

Using the three equations above, we will get the real and imaginary parts, $(ac - bd)$ and $(ad + bc)$ respectively. This requires two real additions.
Real Part:

$$k_3 - k_2 = ac - bd$$

Imaginary Part:

$$k_1 + k_3 = ad + bc$$

(1b) Explain the key concepts behind the Fast Fourier Transform that allow a Discrete Fourier Transform to be computed faster and yet achieve the exact same result.
The key concept behind the Fast Fourier Transform that allows a Discrete Fourier Transform to be computed faster is divide and conquer. This works because W_N is periodic. After all the unique values of W_N are calculated we can reuse those values instead of computing them again. Recall that computing

$$X_k = \sum_{n=0}^{N-1} x[n] W_N^{nk}$$

requires N^2 complex multiplications. If the problem is size N, where N is a power of 2 then the problem can be broken down into two different pieces that require $N^2/4$ complex multiplications each. The problem is broken into even and odd indexed terms, this is called "decimation in time". The total computation required after breaking the signal apart and merging them back together is $N^2/2 + N/2$ multiplications. This is $N^2/2 - N/2$ fewer multiplications than the N^2 multiplications from the DFT.

Problem No. 2: FIR Filter Design

The Hann window function can be written as: $w[n] = [0.5 - 0.5cos(2\pi n/M)]w_R[n]$ where $w_R[n]$ is the rectangular window of length M+1.

(2a) Express the DTFT of w[n] in terms of the DTFT of $w_R[n]$.

$$w[n] = [0.5 - 0.25(e^{j\frac{2\pi n}{M}} + e^{-j\frac{2\pi n}{M}})]w_R[n]$$

$$w[n] = 0.5w_R[n] - 0.25w_R[n]e^{j\frac{2\pi n}{M}}0.25w_R[n]e^{-j\frac{2\pi n}{M}}$$

$$\boxed{W(e^{j\omega}) = 0.5W_R[e^{j\omega}] - 0.25W_R[e^{j(\omega+\frac{2\pi n}{M})}] - 0.25W_R[e^{j(\omega-\frac{2\pi n}{M})}]} \quad (1)$$

(2b) Explain why the Hann window has the wider main lobe but lower side lobes than the rectangular window of the same length.
The Hann window has the wider mainlobe because of the second and third term in equation (1), which shifts its by $\pm 2\pi n/M$ from the origin. Essentially widening the main lobe. The main lobe has a width of $2\pi n/M$. Also the Hann window has lower side lobes because of the scaling factor in equation (1).

(2c) Explain why the width of the main lobe is important. Give an example.
The main lobe is important because it controls the resolution. It is ideal to get a narrow main lobe and very low side lobes. If the width of the main lobe approaches zero, then it will represent an impulse, which will not result in the picket fence effect. The side lobes control aliasing. If the side lobes are high then aliasing will occur. There are many different windows to choose from, each with different trade offs. The Kaiser Window is a general purpose window because it allows control of the trade off between the main lobe width and the highest side lobe level.

Problem No. 3: Let $x[n]$ be an input signal and $h[n]$ denote the causal and stable IIR system. First, $x[n]$ is filtered through $h[n]$ to obtain output $y_1[n]$. Next, the flipped signal, $x[-n]$, is filtered through $h[n]$ to obtain $y_2[n]$. Finally, the output is computed by summing $y[n] = y_1[n]$ and $y_2[n]$.
(3a) Let $h_s[n]$ be the impulse response of the overall system (e.g., $x[n]$ is the input, $y[n]$ is the output). Derive an expression for $h_s[n]$ in terms of $h[n]$.

$$\begin{aligned} y_1[n] &= x[n] * h[n] \\ y_2[n] &= x[-n] * h[n] \end{aligned}$$

$$x[-n] * h[n] = \sum_{m=-\infty}^{\infty} x[-m]h[n-m] = \sum_{m=-\infty}^{\infty} x[m]h[n+m] = x[n] * h[-n]$$

$$\begin{aligned} y[n] &= y_1[n] + y_2[n] = x[n] * h[n] + x[-n] * h[n] \\ &= x[n] * h_s[n] \end{aligned}$$

$$\boxed{h_s[n] = h[n] + h[-n]}$$

(3b) Determine the frequency response of $h_s[n]$ and discuss any important properties you observe.

$$H_s(e^{j\omega}) = H(e^{j\omega}) + H(e^{-j\omega}) = 2Re[H(e^{j\omega})]$$

This is a zero-phase filter because the frequency response $H(e^{j\omega})$ is a real and even function of ω. A function is said to be even when the function is symmetric about the y-axis. Zero-phase filters are non-causal, however this is easily fixed by setting a delay in the filter.

(3c) Will these properties hold if $h[n]$ is an IIR filter? Explain the significance of your findings. Yes, these properties will hold if $h[n]$ is an IIR filter because the result from (3b) is general or doesn't have any restrictions. $H(e^{j\omega})$ can be any type of filter, as long as it has a real part then these property will hold. Zero-phase filters are often preferred when all the data is available and ready to be processed.