Problem No. 1:

1.a. :

$$y[n] = a_1 y[n-1] + a_2 y[n-2] + x[n] + b_1 x[n-1] \xRightarrow{z-transform}$$

$$Y(z) = a_1 z^{-1} Y(z) + a_2 z^{-2} Y(z) + X(z) + b_1 z^{-1} X(z) \Rightarrow$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + b_1 z^{-1}}{1 - a_1 z^{-1} - a_2 z^{-2}} = \frac{z^2 + b_1 z}{z^2 - a_1 z - a_2} \quad (1)$$

We know that a causal LTI with a rational system function is stable if and only if all poles of H(z) are inside the unit circle in the z-plane, or $|p_1| < 1$ and $|p_2| < 1$.The poles of (1) are obtained by solving the quadratic equation $z^2 - a_1 z - a_2 = 0$. The results are:

$$p_{1,2} = \frac{a_1 \pm \sqrt{a_1^2 + 4a_2}}{2} \quad \rightarrow \quad a_1 = p_1 + p_2 \quad \text{and} \quad a_2 = -p_1 p_2$$

So the conditions that a_1 and a_2 must satisfy for stability are:

$$|a_2| = |-p_1 p_2| = |p_1||p_2| < 1$$

The condition for a_1 can be expressed as:

$$|a_1| < 1 - a_2$$

1.b. :

$$H(z) = \frac{1 + b_1 z^{-1}}{1 - a_1 z^{-1} - a_2 z^{-2}}$$

When b_1=0 we have:

$$H(z) = \frac{1}{1 - a_1 z^{-1} - a_2 z^{-2}}$$

The characteristic of the two-pole system depend on the location of the poles or, equivalently, on the values of a_1 and a_2. The poles of the system may be real or complex conjugate, depending on the value of $\Delta = a_1^2 + 4a_2$. So there are three cases:

Real and distinct poles ($a_1^2 > -4a_2$):

Since p_1 *and* p_2 are real and $p_1 \neq p_2$ the system function can be written in this form:

$$H(z) = \frac{A_1}{1 - p_1 z^{-1}} + \frac{A_2}{1 - p_2 z^{-1}}$$

And we have:

$$A_1 = \frac{p_1}{p_1 - p_2} \quad , \quad A_2 = \frac{-p_2}{p_1 - p_2}$$

Then impulse response is:

$$h(n) = \frac{1}{p_1 - p_2}(p_1^{n+1} - p_2^{n+1})u[n]$$

The impulse response is the difference of two decaying exponential sequences. Figure 1.1 illustrates a graph for h[n] when the poles are p_1=0.8 and p2=0.1.

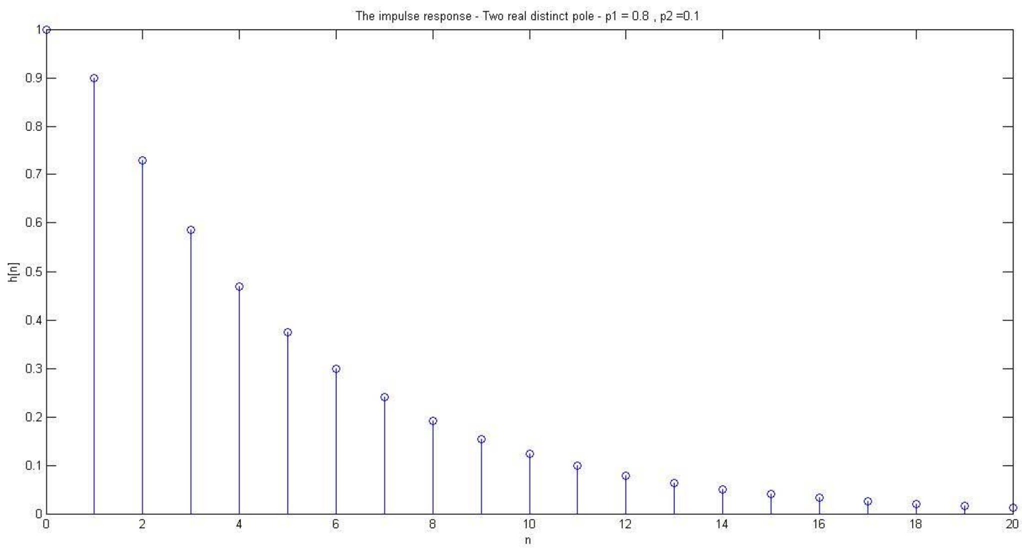


Fig 1.1

Real and equal poles ($a_1^2 = -4a_2$):

In this case we have $p_1 = p_2 = p = a_1/2$. So the system function is:

$$H(z) = \frac{1}{(1 - pz^{-1})^2}$$

And the impulse response is:

$$h[n] = (n + 1)p^n u[n]$$

In this situation h[n] is the product of a ramp sequence and a real decaying exponential sequence. Figure 1.2 illustrates a graph for h[n] when the poles are p1=p2=0.1.

Complex-Conjugate poles ($a_1^2 < -4a_2$):

As the poles are complex-conjugate the system function can be written in this form:

$$H(z) = \frac{A}{1 - pz^{-1}} + \frac{A^*}{1 - p^*z^{-1}} = \frac{A}{1 - re^{j\omega_0}z^{-1}} + \frac{A^*}{1 - re^{-j\omega_0}z^{-1}}$$

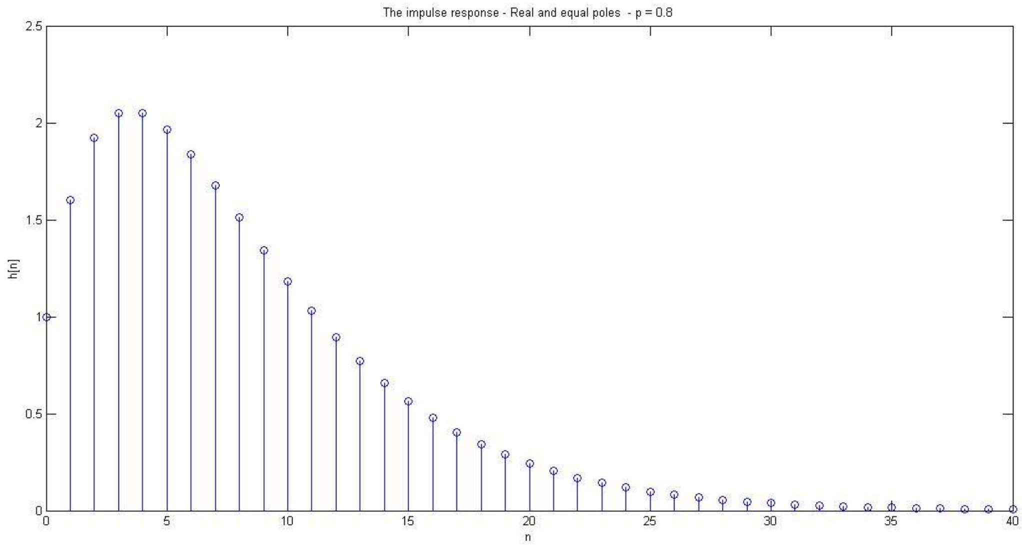


Fig 1.2

And we have

$p = re^{j\omega} \qquad 0 < \omega_0 < \pi$

As the poles are complex conjugate, the parameters a_1 and a_2 are related to r and ω_0 according to

$a_1 = 2r\cos\omega_0$

$a_2 = r^2$

And also the parameter of A in partial fraction equals to:

$$A = \frac{p}{p - p^*} = \frac{re^{j\omega_0}}{r(e^{j\omega_0} - e^{-j\omega_0})} = \frac{e^{j\omega_0}}{j2\sin\omega_0}$$

So impulse response equals to:

$$h[n] = \frac{r^n}{\sin\omega_0}\sin[n+1]\,\omega_0 u[n]$$

When r<1 then h[n] is oscillating with an exponentially decaying envelope. In this situation, the angle ω_0 of the poles determines the frequency of oscillation and the distance r of the poles from the origin determines the rate of decay (Fig 1.3).

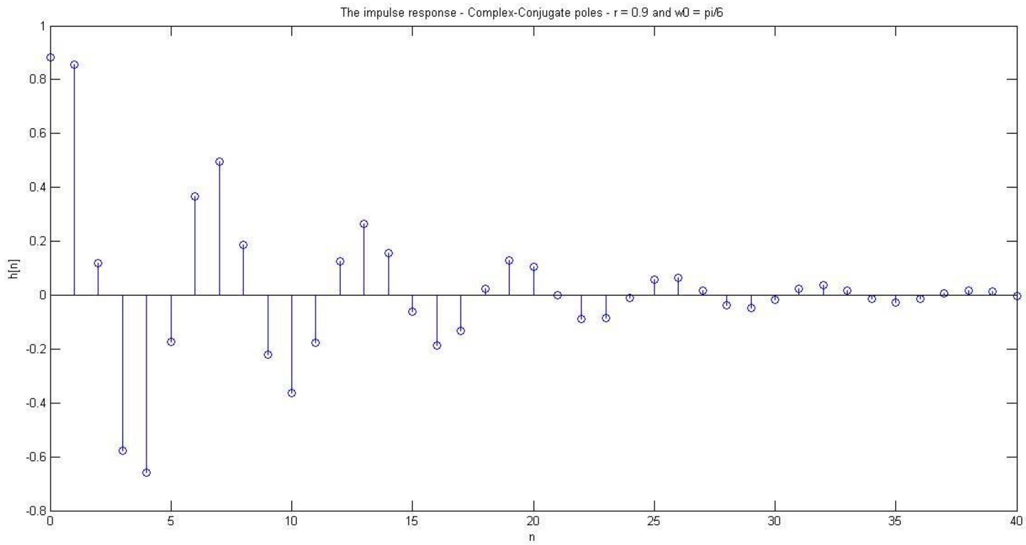


Fig (1.3)

When r becomes close to unit circle the decaying is slow. When it is on the unit circle it is oscillating (Fig1.4).

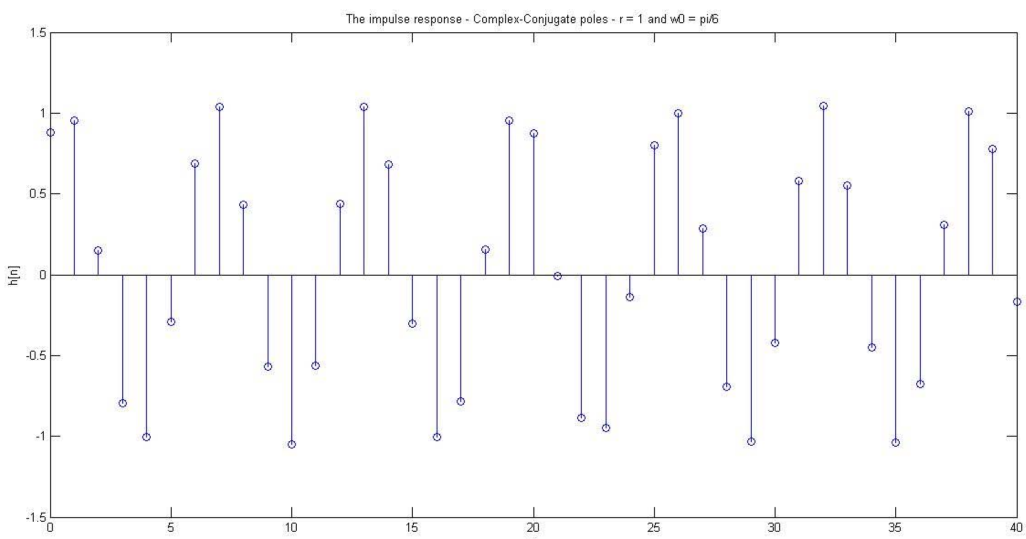


Fig (1.4)

When it is outside the unit circle the magnitude of oscillation is increasing (Fig 1.5).

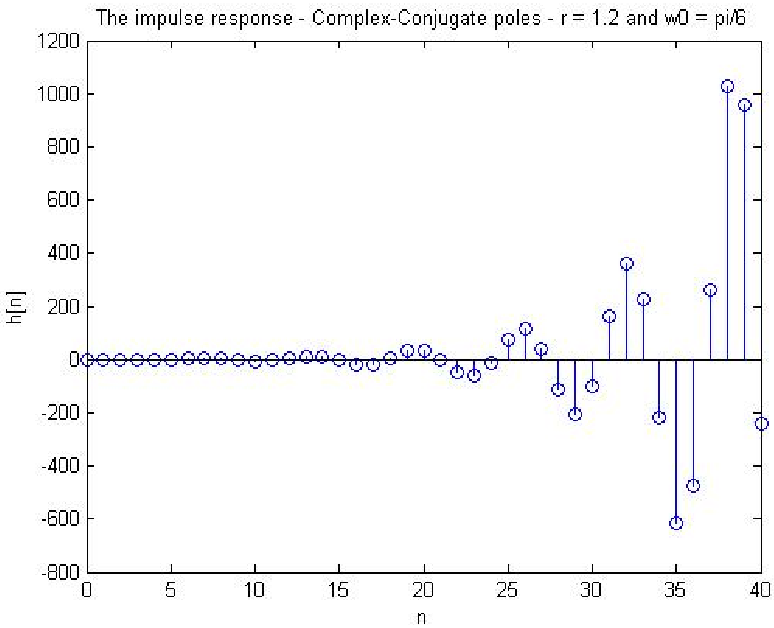


Fig 1.5.

(c)

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + b_1 z^{-1}}{1 - a_1 z^{-1} - a_2 z^{-2}} = \frac{z^2 + b_1 z}{z^2 - a_1 z - a_2}$$

The zeroes of the system function are z=0 and z=-b_1. The zeros of the system function introduce a constant scaling factor and a constant phase shift. It is also create valleys in the plot. A plot of the locations of the poles and zeroes in the z-plane for b_1 = 0.9, a_1= 0.4 and a_2=-0.8 and its impulse response are illustrated in 1.6.

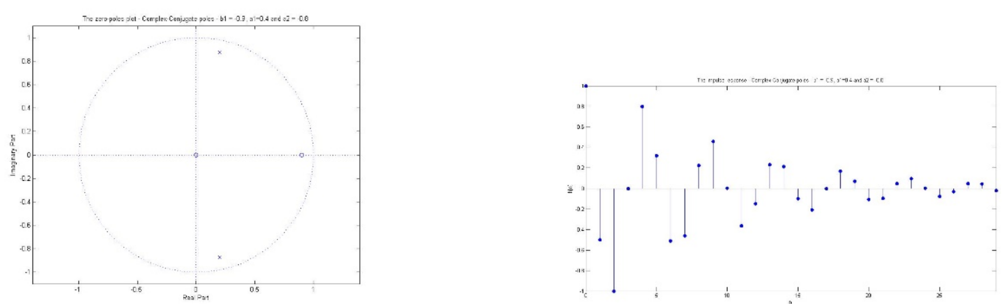


Fig (1.6)

As it is explained in 1.b when poles are close to unit circle the exponential is decaying slow and when the poles are close to origin the exponential is decaying fast. This behavior is depicted in 1.7 in comparison with 1.6. As you can see the poles are closer to origin and the exponential is decaying faster.

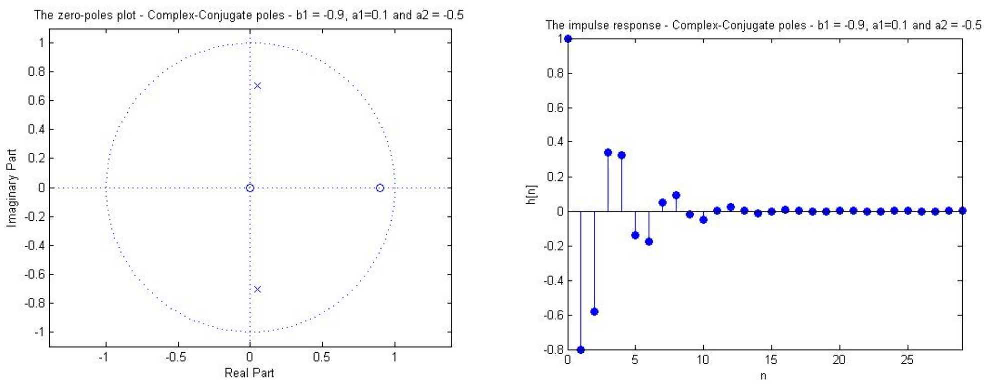


Fig (1.7)

Problem No. 2:

2.a. :

$x_1[n]=0.5x[n-1]$ =>

$x_2[n]=x[n] + x_1[n]=x[n] + 0.5x[n-1]$ =>

$x_3[n]=0.25x_1[n-1]=0.125x[n-2]$ =>

$y[n] = x_2[n] + x3[n]$ =>

$y[n] = x[n] + 0.5x[n-1]+ 0.125x[n-2]$ (1)

Convolution Definition:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]. \quad -\infty < n < \infty \quad =>$$

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]. \quad -\infty < n < \infty \quad (2)$$

From (1) and (2):

h[0]=1, h[1]=1/2,h[2]=1/8 =>

h[n]=[1 1/2 1/8]

2.b. :

z-transform of (1) =>

$$Y(z) = X(z) + \frac{1}{2}z^{-1}X(z) + \frac{1}{8}z^{-2}X(z) \Rightarrow$$

$$H(z) = \frac{Y(z)}{X(z)} = 1 + \frac{1}{2}z^{-1} + \frac{1}{8}z^{-2}$$

Problem No. 3:

3.a. :

$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$ =>

$X(e^{j\omega}) = e^{-j\omega} + 2e^{-3j\omega}$ =>

$$X(e^{j\omega}) = e^{-j\omega} + 2e^{-3j\omega} \qquad (1)$$

$$\omega = 2\pi f = 2\pi \frac{F}{F_s} = \pi F \qquad (2)$$

From (1) and (2):

$X(F) = e^{-j\pi F} + 2e^{-3j\pi F}$

The Fourier transform of X which is periodic with fundamental frequency of 2 Hz is shown in Fig 3.1.

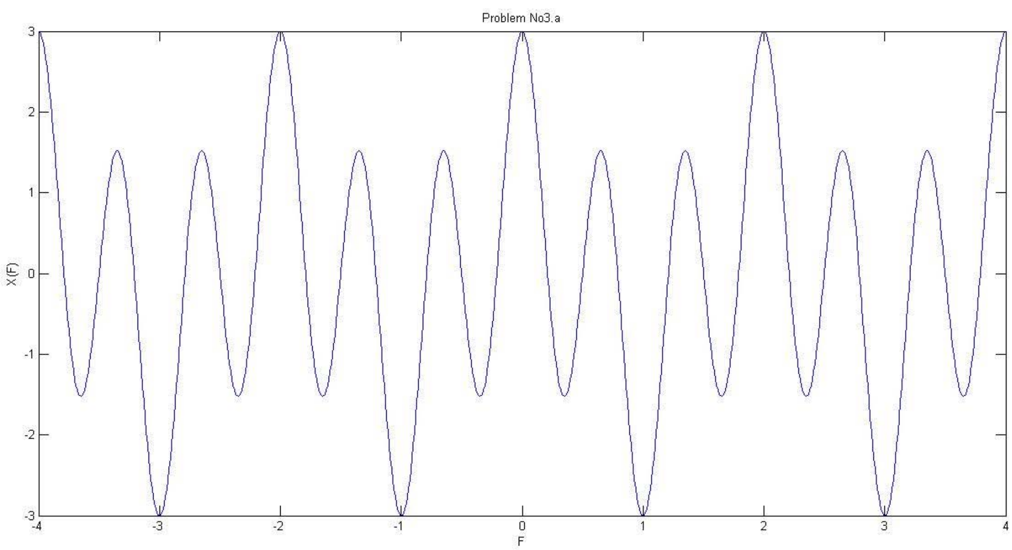


Fig. 3.1

3.b.

For discrete and periodic signals DTFS equals:

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}kn} \quad =>$$

$$c_k = \frac{1}{10}\sum_{n=0}^{9} x[n]e^{-j\frac{2\pi}{10}kn} \quad =>$$

For example we have:

$$c_0 = \frac{3}{10}$$

$$c_1 = \frac{1}{10}\left(e^{-j\frac{2\pi}{10}} + 2e^{-j\frac{6\pi}{10}}\right)$$

$$c_2 = \frac{1}{10}\left(e^{-j\frac{4\pi}{10}} + e^{-j\frac{12\pi}{10}}\right)$$

$$c_3 = \frac{1}{10}\left(e^{-j\frac{6\pi}{10}} + e^{-j\frac{18\pi}{10}}\right)$$

.

.

.

$$c_9 = \frac{1}{10}\left(e^{-j\frac{18\pi}{10}} + e^{-j\frac{54\pi}{10}}\right)$$

Or

c = [0.3000 0.0191 - 0.2490i -0.1309 + 0.0225i 0.1309 + 0.0225i -0.0191 - 0.2490i

-0.3000 -0.0191 + 0.2490i 0.1309 - 0.0225i -0.1309 - 0.0225i 0.0191 + 0.2490i]

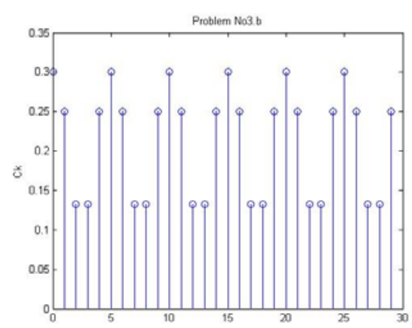


Fig 3.2

The plot of DTFS of x[n] is illustrated in Fig 3.2. As you can see the DTFT is continuous and periodic signal but DTFS is discrete and periodic signal.

3.c.

For discrete and periodic signals DTFS equals:

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}kn} \quad =>$$

$$c_k = \frac{1}{5}\sum_{n=0}^{4} x[n]e^{-j\frac{2\pi}{5}kn} \quad =>$$

For example we have:

$$c_0 = \frac{3}{5}$$

$$c_1 = \frac{1}{5}\left(e^{-j\frac{2\pi}{5}} + 2e^{-j\frac{6\pi}{5}}\right)$$

$$c_2 = \frac{1}{5}\left(e^{-j\frac{4\pi}{5}} + e^{-j\frac{12\pi}{5}}\right)$$

$$c_3 = \frac{1}{5}\left(e^{-j\frac{6\pi}{5}} + e^{-j\frac{18\pi}{5}}\right)$$

$$c_4 = \frac{1}{5}\left(e^{-j\frac{18\pi}{5}} + e^{-j\frac{54\pi}{5}}\right)$$

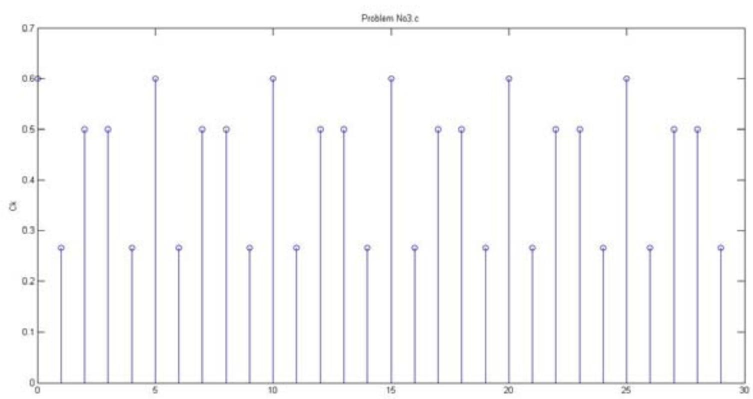


Fig 3.3

The plot of DTFS of x[n] is illustrated in Fig 3.3. As you can see the DTFT is continuous and periodic signal but we should consider the fact that in this case the period of signal is N=5. However in 3.b the period of signal is N=10.

Problem No. 4:

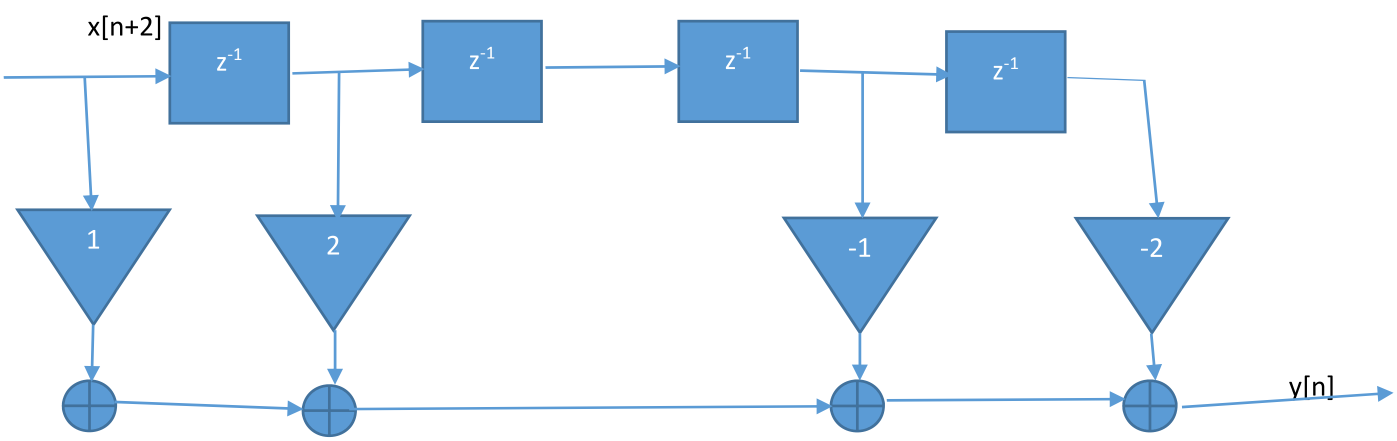
(a)

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]. \quad -\infty < n < \infty \quad =>$$

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]. \quad -\infty < n < \infty$$

$$y[n] = h[-2]x[n+2] + h[-1]x[n+1] + h[0]x[n] + h[1]x[n-1] + h[2]x[n-2] \quad =>$$

$$y[n] = x[n+2] + 2x[n+1] - x[n-1] - 2x[n-2] \quad =>$$



(b)

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]. \quad -\infty < n < \infty \quad =>$$

Folding x[n] and sliding through h[n] =>

h[n]					1	-1						
h[n]	-2	-1	0	2	1							=1
		-2	-1	0	2	1						=1
			-2	-1	0	2	1					=-2
				-2	-1	0	2	1				=-1
					-2	-1	0	2	1			=-1
						-2	-1	0	2	1		=2

=>

y[n] = [1 1 -2 -1 -1 2]