## B.7-5 Complex Numbers

$$e^{\pm j\pi/2} = \pm j$$

$$e^{\pm jn\pi} = \begin{cases} 1 & n \text{ even} \\ -1 & n \text{ odd} \end{cases}$$

$$e^{\pm j\theta} = \cos\theta \pm j\sin\theta$$

$$a + jb = re^{j\theta} \qquad r = \sqrt{a^2 + b^2}. \qquad \theta = \tan^{-1}\left(\tfrac{b}{a}\right)$$

$$(re^{j\theta})^k = r^k e^{jk\theta}$$

$$(r_1 e^{j\theta_1})(r_2 e^{j\theta_2}) = r_1 r_2 e^{j(\theta_1+\theta_2)}$$

## B.7-6 Trigonometric Identities

$$e^{\pm jx} = \cos x \pm j\sin x$$

$$\cos x = \tfrac{1}{2}[e^{jx} + e^{-jx}]$$

$$\sin x = \tfrac{1}{2j}[e^{jx} - e^{-jx}]$$

$$\cos\left(x \pm \tfrac{\pi}{2}\right) = \mp\sin x$$

$$\sin\left(x \pm \tfrac{\pi}{2}\right) = \pm\cos x$$

$$2\sin x\cos x = \sin 2x$$

$$\sin^2 x + \cos^2 x = 1$$

$$\cos^2 x - \sin^2 x = \cos 2x$$

$$\cos^2 x = \tfrac{1}{2}(1 + \cos 2x)$$

$$\sin^2 x = \tfrac{1}{2}(1 - \cos 2x)$$

$$\cos^3 x = \tfrac{1}{4}(3\cos x + \cos 3x)$$

$$\sin^3 x = \tfrac{1}{4}(3\sin x - \sin 3x)$$

$$\sin(x \pm y) = \sin x\cos y \pm \cos x\sin y$$

$$\cos(x \pm y) = \cos x\cos y \mp \sin x\sin y$$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x\tan y}$$

$$\sin x\sin y = \tfrac{1}{2}[\cos(x - y) - \cos(x + y)]$$

$$\cos x\cos y = \tfrac{1}{2}[\cos(x - y) + \cos(x + y)]$$

$$\sin x\cos y = \tfrac{1}{2}[\sin(x - y) + \sin(x + y)]$$

$$a\cos x + b\sin x = C\cos(x + \theta)$$

$$\text{in which } C = \sqrt{a^2 + b^2} \quad \text{and} \quad \theta = \tan^{-1}\left(\frac{-b}{a}\right)$$

**Table 4.1**

**A Short Table of Fourier Transforms**

| | $f(t)$ | $F(\omega)$ | |
|---|---|---|---|
| 1 | $e^{-at}u(t)$ | $\frac{1}{a+j\omega}$ | $a > 0$ |
| 2 | $e^{at}u(-t)$ | $\frac{1}{a-j\omega}$ | $a > 0$ |
| 3 | $e^{-a\|t\|}$ | $\frac{2a}{a^2+\omega^2}$ | $a > 0$ |
| 4 | $te^{-at}u(t)$ | $\frac{1}{(a+j\omega)^2}$ | $a > 0$ |
| 5 | $t^n e^{-at}u(t)$ | $\frac{n!}{(a+j\omega)^{n+1}}$ | $a > 0$ |
| 6 | $\delta(t)$ | $1$ | |
| 7 | $1$ | $2\pi\delta(\omega)$ | |
| 8 | $e^{j\omega_0 t}$ | $2\pi\delta(\omega-\omega_0)$ | |
| 9 | $\cos \omega_0 t$ | $\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$ | |
| 10 | $\sin \omega_0 t$ | $j\pi[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)]$ | |
| 11 | $u(t)$ | $\pi\delta(\omega)+\frac{1}{j\omega}$ | |
| 12 | $\operatorname{sgn} t$ | $\frac{2}{j\omega}$ | |
| 13 | $\cos \omega_0 t\, u(t)$ | $\frac{\pi}{2}[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]+\frac{j\omega}{\omega_0^2-\omega^2}$ | |
| 14 | $\sin \omega_0 t\, u(t)$ | $\frac{\pi}{2j}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]+\frac{\omega_0}{\omega_0^2-\omega^2}$ | |
| 15 | $e^{-at}\sin \omega_0 t\, u(t)$ | $\frac{\omega_0}{(a+j\omega)^2+\omega_0^2}$ | $a > 0$ |
| 16 | $e^{-at}\cos \omega_0 t\, u(t)$ | $\frac{a+j\omega}{(a+j\omega)^2+\omega_0^2}$ | $a > 0$ |
| 17 | $\operatorname{rect}\left(\frac{t}{\tau}\right)$ | $\tau \operatorname{sinc}\left(\frac{\omega\tau}{2}\right)$ | |
| 18 | $\frac{W}{\pi}\operatorname{sinc}(Wt)$ | $\operatorname{rect}\left(\frac{\omega}{2W}\right)$ | |
| 19 | $\Delta\left(\frac{t}{\tau}\right)$ | $\frac{\tau}{2}\operatorname{sinc}^2\left(\frac{\omega\tau}{4}\right)$ | |
| 20 | $\frac{W}{2\pi}\operatorname{sinc}^2\left(\frac{Wt}{2}\right)$ | $\Delta\left(\frac{\omega}{2W}\right)$ | |
| 21 | $\sum_{n=-\infty}^{\infty}\delta(t-nT)$ | $\omega_0\sum_{n=-\infty}^{\infty}\delta(\omega-n\omega_0)$ | $\omega_0=\frac{2\pi}{T}$ |
| 22 | $e^{-t^2/2\sigma^2}$ | $\sigma\sqrt{2\pi}e^{-\sigma^2\omega^2/2}$ | |

**Table 4.2**

**Fourier Transform Operations**

| Operation | $f(t)$ | $F(\omega)$ |
|---|---|---|
| Addition | $f_1(t) + f_2(t)$ | $F_1(\omega) + F_2(\omega)$ |
| Scalar multiplication | $kf(t)$ | $kF(\omega)$ |
| Symmetry | $F(t)$ | $2\pi f(-\omega)$ |
| Scaling ($a$ real) | $f(at)$ | $\frac{1}{\lvert a\rvert} F\left(\frac{\omega}{a}\right)$ |
| Time shift | $f(t - t_0)$ | $F(\omega)e^{-j\omega t_0}$ |
| Frequency shift ($\omega_0$ real) | $f(t)e^{j\omega_0 t}$ | $F(\omega - \omega_0)$ |
| Time convolution | $f_1(t) * f_2(t)$ | $F_1(\omega)F_2(\omega)$ |
| Frequency convolution | $f_1(t)f_2(t)$ | $\frac{1}{2\pi} F_1(\omega) * F_2(\omega)$ |
| Time differentiation | $\frac{d^n f}{dt^n}$ | $(j\omega)^n F(\omega)$ |
| Time integration | $\int_{-\infty}^{t} f(x)\,dx$ | $\frac{F(\omega)}{j\omega} + \pi F(0)\delta(\omega)$ |

Repeated application of this property yields

$$\frac{d^n f}{dt^n} \Longleftrightarrow (j\omega)^n F(\omega) \tag{4.48}$$

The time-integration property [Eq. (4.47)] has already been proved in Example 4.13.

■ **Example 4.14**

Using the time-differentiation property, find the Fourier transform of the triangle pulse $\Delta(\frac{t}{\tau})$ illustrated in Fig. 4.25a.

To find the Fourier transform of this pulse we differentiate the pulse successively, as illustrated in Fig. 4.25b and c. Because $df/dt$ is constant everywhere, its derivative, $d^2f/dt^2$, is zero everywhere. But $df/dt$ has jump discontinuities with a positive jump of $2/\tau$ at $t = \pm\frac{\tau}{2}$, and a negative jump of $4/\tau$ at $t = 0$. Recall that the derivative of a signal at a jump discontinuity is an impulse at that point of strength equal to the amount of jump. Hence, $d^2f/dt^2$, the derivative of $df/dt$, consists of a sequence of impulses, as depicted in Fig. 4.25c; that is,

$$\frac{d^2 f}{dt^2} = \frac{2}{\tau}\left[\delta\left(t + \frac{\tau}{2}\right) - 2\delta(t) + \delta\left(t - \frac{\tau}{2}\right)\right] \tag{4.49}$$

## Table 4.3

## Some Window Functions and Their Characteristics

| | Window $w(t)$ | Mainlobe Width | Rolloff Rate dB/oct | Peak Sidelobe Level in dB |
|---|---|---|---|---|
| 1 | Rectangular: $\mathrm{rect}(\frac{t}{T})$ | $\frac{4\pi}{T}$ | $-6$ | $-13.3$ |
| 2 | Bartlett: $\Delta(\frac{t}{2T})$ | $\frac{8\pi}{T}$ | $-12$ | $-26.5$ |
| 3 | Hanning: $0.5\left[1+\cos\left(\frac{2\pi t}{T}\right)\right]$ | $\frac{8\pi}{T}$ | $-18$ | $-31.5$ |
| 4 | Hamming: $0.54+0.46\cos\left(\frac{2\pi t}{T}\right)$ | $\frac{8\pi}{T}$ | $-6$ | $-42.7$ |
| 5 | Blackman: $0.42+0.5\cos\left(\frac{2\pi t}{T}\right)+0.08\cos\left(\frac{4\pi t}{T}\right)$ | $\frac{12\pi}{T}$ | $-18$ | $-58.1$ |
| 6 | Kaiser: $\dfrac{I_0\left[\alpha\sqrt{1-4(\frac{t}{T})^2}\right]}{I_0(\alpha)}$ $\quad 1\le\alpha\le 10$ | $\frac{11.2\pi}{T}$ | $-6$ | $-59.9$ $(\alpha=8.168)$ |

**Table 6.2**

**The Laplace Transform Properties**

| Operation | $f(t)$ | $F(s)$ |
|---|---|---|
| Addition | $f_1(t) + f_2(t)$ | $F_1(s) + F_2(s)$ |
| Scalar multiplication | $kf(t)$ | $kF(s)$ |
| Time differentiation | $\dfrac{df}{dt}$ | $sF(s) - f(0^-)$ |
| | $\dfrac{d^2f}{dt^2}$ | $s^2F(s) - sf(0^-) - \dot{f}(0^-)$ |
| | $\dfrac{d^3f}{dt^3}$ | $s^3F(s) - s^2f(0^-) - s\dot{f}(0^-) - \ddot{f}(0^-)$ |
| Time integration | $\int_{0^-}^{t} f(\tau)\,d\tau$ | $\dfrac{1}{s}F(s)$ |
| | $\int_{-\infty}^{t} f(\tau)\,d\tau$ | $\dfrac{1}{s}F(s) + \dfrac{1}{s}\int_{-\infty}^{0^-} f(t)\,dt$ |
| Time shift | $f(t - t_0)u(t - t_0)$ | $F(s)e^{-st_0}\quad t_0 \geq 0$ |
| Frequency shift | $f(t)e^{s_0 t}$ | $F(s - s_0)$ |
| Frequency differentiation | $-tf(t)$ | $\dfrac{dF(s)}{ds}$ |
| Frequency integration | $\dfrac{f(t)}{t}$ | $\int_{s}^{\infty} F(z)\,dz$ |
| Scaling | $f(at),\ a \geq 0$ | $\dfrac{1}{a}F\left(\dfrac{s}{a}\right)$ |
| Time convolution | $f_1(t) * f_2(t)$ | $F_1(s)F_2(s)$ |
| Frequency convolution | $f_1(t)f_2(t)$ | $\dfrac{1}{2\pi j}F_1(s) * F_2(s)$ |
| Initial value | $f(0^+)$ | $\lim_{s\to\infty} sF(s)\qquad (n > m)$ |
| Final value | $f(\infty)$ | $\lim_{s\to 0} sF(s)\quad$ (poles of $sF(s)$ in LHP) |

**Table 6.1**

**A Short Table of (Unilateral) Laplace Transforms**

| | $f(t)$ | $F(s)$ |
|---|---|---|
| 1 | $\delta(t)$ | $1$ |
| 2 | $u(t)$ | $\frac{1}{s}$ |
| 3 | $tu(t)$ | $\frac{1}{s^2}$ |
| 4 | $t^n u(t)$ | $\frac{n!}{s^{n+1}}$ |
| 5 | $e^{\lambda t}u(t)$ | $\frac{1}{s-\lambda}$ |
| 6 | $te^{\lambda t}u(t)$ | $\frac{1}{(s-\lambda)^2}$ |
| 7 | $t^n e^{\lambda t}u(t)$ | $\frac{n!}{(s-\lambda)^{n+1}}$ |
| 8$a$ | $\cos bt\, u(t)$ | $\frac{s}{s^2+b^2}$ |
| 8$b$ | $\sin bt\, u(t)$ | $\frac{b}{s^2+b^2}$ |
| 9$a$ | $e^{-at}\cos bt\, u(t)$ | $\frac{s+a}{(s+a)^2+b^2}$ |
| 9$b$ | $e^{-at}\sin bt\, u(t)$ | $\frac{b}{(s+a)^2+b^2}$ |
| 10$a$ | $re^{-at}\cos(bt+\theta)\, u(t)$ | $\frac{(r\cos\theta)s+(ar\cos\theta-br\sin\theta)}{s^2+2as+(a^2+b^2)}$ |
| 10$b$ | $re^{-at}\cos(bt+\theta)\, u(t)$ | $\frac{0.5re^{j\theta}}{s+a-jb}+\frac{0.5re^{-j\theta}}{s+a+jb}$ |
| 10$c$ | $re^{-at}\cos(bt+\theta)\, u(t)$ | $\frac{As+B}{s^2+2as+c}$ |
| | $r=\sqrt{\frac{A^2c+B^2-2ABa}{c-a^2}},\ \theta=\tan^{-1}\frac{Aa-B}{A\sqrt{c-a^2}}$ | |
| | $b=\sqrt{c-a^2}$ | |
| 10$d$ | $e^{-at}\left[A\cos bt+\frac{B-Aa}{b}\sin bt\right]u(t)$ | $\frac{As+B}{s^2+2as+c}$ |
| | $b=\sqrt{c-a^2}$ | |