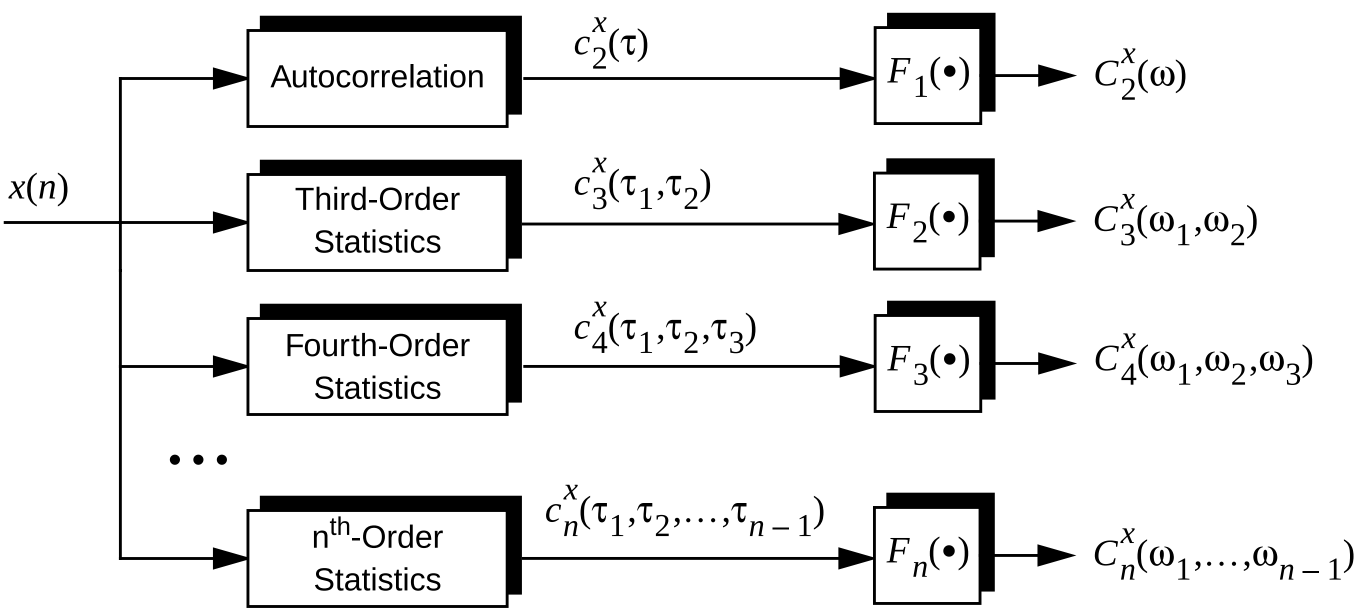
Higher-Order Spectra

- Fact: the information contained in the power spectrum of a signal is essentially that which is contained in the autocorrelation sequence
- This is sufficient for a signal with Gaussian statistics (why?), but...
- Premise: there is useful information in higher-order moments of the signal (typically third and fourth-order moments or cumulants)

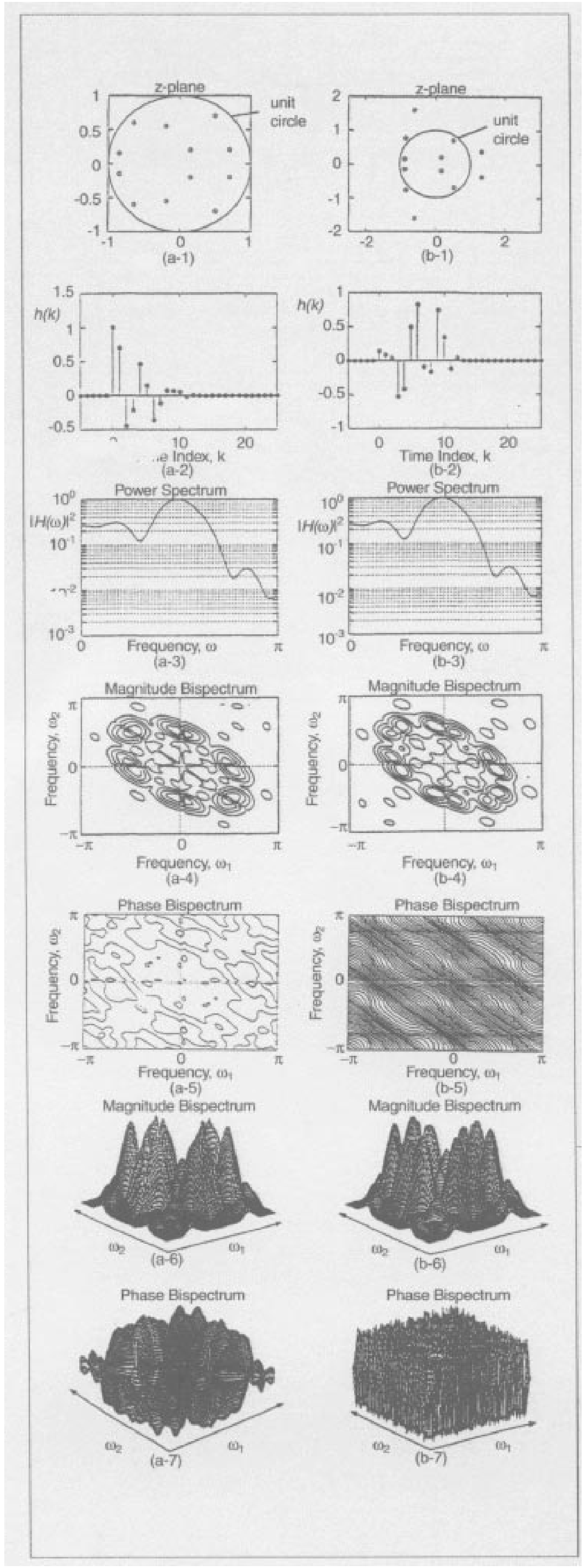
Historical Perspective:

- The power spectrum is a member of the class of higher order spectra, and can be referred to as a second-order spectra:



- Such transformations can suppress additive colored Gaussian noise of unknown power spectra
- Identify and/or reconstruct non-minimum phase signals
- Characterize deviations from Gaussian statistics
- Identify nonlinear properties in signals

An Example: Non-Minimum Phase Systems



Mathematical Definitions

The nth-order moment is defined as:

$$m_n^x(\tau_1,\tau_2,\ldots,\tau_{n-1}) \equiv E[x(n)x(n+\tau_1)x(n+\tau_2)\ldots x(n+\tau_{n-1})]$$

The nth-order cumulant is defined as:

$$c_n^x(\tau_1,\tau_2,\ldots,\tau_{n-1}) \equiv m_n^x(\tau_1,\tau_2,\ldots,\tau_{n-1}) - m_n^G(\tau_1,\tau_2,\ldots,\tau_{n-1})$$

where $m_n^G(\tau_1,\tau_2,\ldots,\tau_{n-1})$ is the nth-order moment of an equivalent Gaussian signal with the same mean and autocorrelation as $x(n)$.

We can expand these functions as follows:

First-order moment:

$$c_1^x = m_1^x = E[x(n)] \qquad \text{(mean value)}$$

Second-order moment:

$$\begin{aligned} c_2^x(\tau_1) &= m_2^x(\tau_1) - (m_1^x)^2 \\ &= m_2^x(-\tau_1) - (m_1^x)^2 \qquad \text{(covariance)} \\ &= c_2^x(\tau_1) \end{aligned}$$

Third-order moment:

$$c_3^x(\tau_1,\tau_2) = m_3^x(\tau_1,\tau_2) - m_1^x[m_2^x(\tau_1) + m_2^x(\tau_2) + m_2^x(\tau_1-\tau_2)] + 2(m_1^x)^2$$

If the mean of the signal is zero, then the second and third-order cumulants are equivalent to the moments, and the fourth-order cumulant is given by:

$$\begin{aligned} c_4^x(\tau_1,\tau_2,\tau_3) = {} & m_4^x(\tau_1,\tau_2,\tau_3) - m_2^x(\tau_1)m_2^x(\tau_3-\tau_2) - \\ & m_2^x(\tau_2)m_2^x(\tau_3-\tau_1) - m_3^x(\tau_3)m_2^x(\tau_2-\tau_1) \end{aligned}$$

Mathematical Properties of Deterministic Signals

- Spectrum

Energy spectrum: $M_2^x(\omega) = X(\omega)X^*(\omega)$

Bispectrum: $M_3^x(\omega_1, \omega_2) = X(\omega_1)X^*(\omega_2)X^*(\omega_1 + \omega_2)$

Trispectrum: $M_4^x(\omega_1, \omega_2, \omega_3) = X(\omega_1)X(\omega_2)X(\omega_3)X^*(\omega_1 + \omega_2 + \omega_3)$

- Quality Measures (zero mean)

Variance: $\gamma_2^x = E[x^2(n)] = c_2^x(0)$

Skewness: $\gamma_3^x = E[x^3(n)] = c_3^x(0, 0)$

Kurtosis: $\gamma_4^x = E[x^4(n)] - 3[\gamma_2^x]^2 = c_4^x(0, 0, 0)$

Normalized Kurtosis: $\gamma_4^x / [\gamma_2^x]^2$

- Coherency

Biocoherency: $$P_3^x(\omega_1,\omega_2) = \frac{C_3^x(\omega_1,\omega_2)}{\sqrt{C_2^x(\omega_1)C_2^x(\omega_2)C_2^x(\omega_1 + \omega_2)}}$$

Tricoherency: $$P_4^x(\omega_1,\omega_2,\omega_3) = \frac{C_3^x(\omega_1,\omega_2)}{\sqrt{C_2^x(\omega_1)C_2^x(\omega_2)C_3^x(\omega_3)C_2^x(\omega_1 + \omega_2 + \omega_3)}}$$

The latter measures are useful in discriminating linear processes from nonlinear ones. A signal is said to be a linear non-Gaussian process of order n if the magnitude of the n^{th}-order coherency function is constant for all frequencies.