Wavelets and Time-Frequency Analysis

- ❒ Premise: signal is decomposed in terms of dilates and translates of a SINGLE function (the mother wavelet)
- ❒ This transform has a number of useful properties (e.g., linearity) and an ability to trade time and frequency resolution in a controlled manner.
- ❒ The wavelet transform provides compact representations of a wide class of deterministic and stochastic signals.

Historical Perspective (originally introduced in the late 1930s):

- ❒ Signal representation as a countably infinite set of basis functions (discrete transforms such as the Fourier series or Karhunen-Loeve representation of stochastic processes) or as a weighted integral of a particular function (continuous transforms such as the Fourier transform)
- ❒ Such transforms are not “localized in time”
- ❒ Long windows imply high frequency resolution, short windows imply low frequency resolution — can we trade resolution in both domains?
- ❒ Can we create a transform that consists of an analysis of the signal at many time scales simultaneously?

Wavelet Overview:

- ❒ Two forms: continuous and discrete wavelet transforms

  The continuous wavelet uses arbitrary dilations and translations; the discrete wavelet encodes these into a parameter that takes discrete values.
- ❒ Both are continuous time signal representations; discrete time representations exist in each case.

The Continuous Wavelet Transform

Assuming $f(t)$ has finite energy,

$$f(t) = \frac{1}{C_\psi} \int_{-\infty}^{\infty} \int_{0}^{\infty} \sqrt{s} F(s, u) \psi(s(t-u)) du ds$$

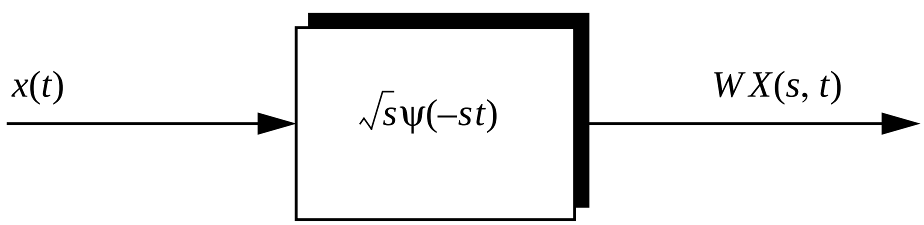
where C_ψ is a finite constant and $F(s, u)$ is the wavelet transform of $f(t)$:

$$F(s, u) = \sqrt{s} \int_{-\infty}^{\infty} f(t) \psi(s(t-u)) dt$$

The variable s is the scale variable because it controls the effective width of support of $\psi(t)$. The variable u has the dimension of time and controls the amount of translation of $\psi(st)$.

The wavelet transform obeys a number of important properties including linearity, superposition, similarity (scaling) and shifting.

It has a simple systems interpretation:



The wavelet transform produces a decomposition of $f(t)$ in terms of a filter bank that consists of filters with impulse responses derived from a single impulse response, $\psi(-t)$.

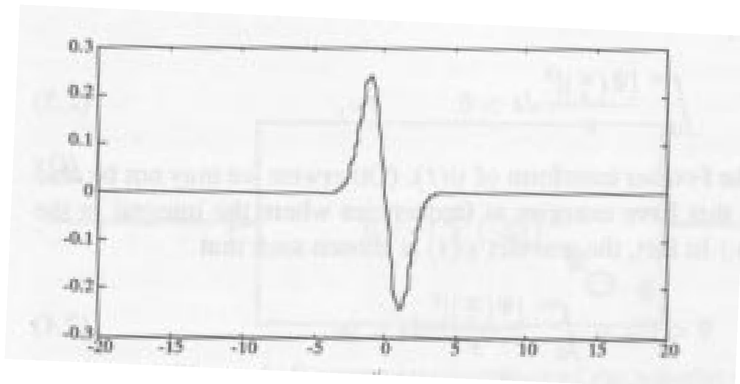
The wavelet transform is energy preserving:

$$\int_{-\infty}^{\infty} \int_{0}^{\infty} |F(s, u)|^2 ds du = C_\psi \int_{-\infty}^{\infty} |f(t)|^2 dt$$

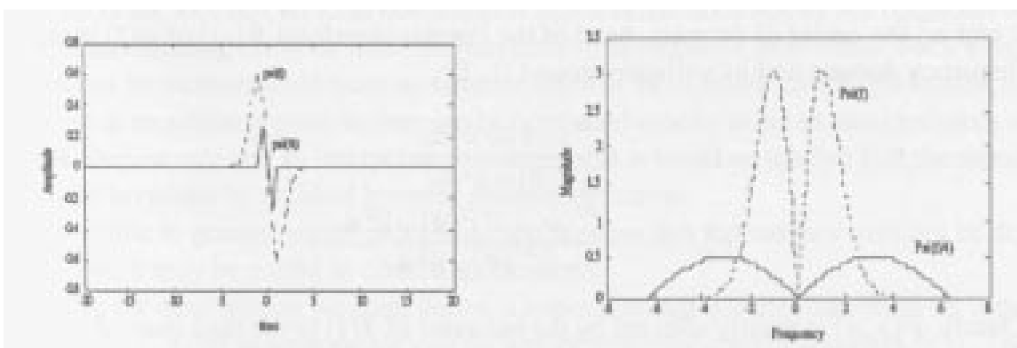
It can also be shown that for smooth signals, most of the energy in $F(s, u)$ will appear at lower scales (an important practical consideration).

An Example of a Wavelet — Second Derivative of a Gaussian

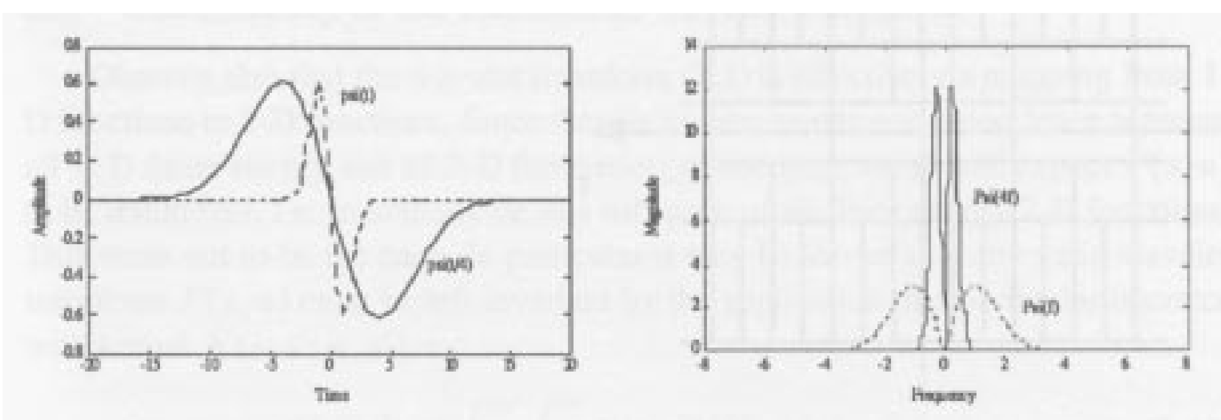
The “mother wavelet”:



Time compression:



Time expansion:



Discrete Orthogonal Wavelet Transform

The continuous transform is highly redundant. An approach to eliminating this is to sample the scale parameter, s, on a grid $\{a^j\}\Big|_{j=-\infty}^{\infty}$. This gives rise to the following transform pair:

$$f(t) = \sum_j \sum_m \sqrt{2^j}\, b(j;m)\psi(2^j t - m)$$

$$b(j;m) = \sqrt{2^j} \int_{-\infty}^{\infty} f(t)\psi(2^j t - m)dt$$

Mathematically, we can construct a discrete wavelet by solving a two-scale difference equation (a dilation equation):

$$\phi(t) = \sum_k c_k \phi(2t - k)$$

where $\phi(t)$ satisfies the following constraints:

$\int \phi(t)dt = 1$, which implies, $\sum_{k=0}^{K-1} c_k = 2$,

The wavelet is constructed from $\phi(t)$ as:

$$\psi(t) = \sum_k d_k \phi(2t - k),$$

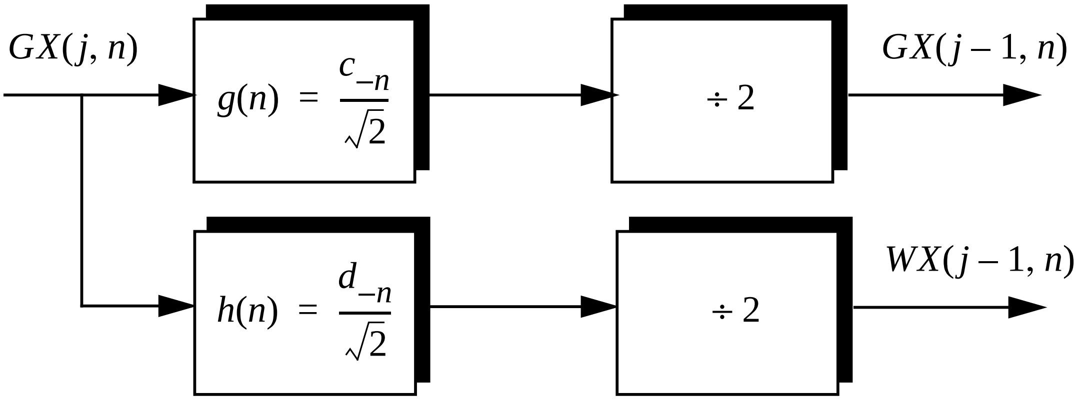
where $d_k = (-1)^k c_{1-k}$.

Note that $\phi(2^j t - m)$ is orthogonal to $\psi(2^j t - n)$ for $j \le m$. We can compute the Fourier transform of c_k:

$$C(\omega) \equiv \frac{1}{2} \sum_k c_k e^{-jk\omega}$$

and show that $\Phi(2\omega) = F\{\phi(2^j t)\} = G(\omega)\Phi(\omega)$.

This gives rise to the following important practical implementation of the wavelet transform:



Mathematically, this can be expressed as:

$$a(j-1;m) \equiv \frac{1}{\sqrt{2}} \sum_{k} c_{k-2m} a(j;k)$$

and

$$b(j-1;m) \equiv \frac{1}{\sqrt{2}} \sum_{k} d_{k-2m} a(j;k)$$

With finite data sets, it is not possible to compute these equations exactly. Hence, we assume that the signal is periodic outside the window (other extensions are possible).

Much like the DFT can be computed using a recursive matrix formulation, the DWT can also be computed using a matrix operation of complexity proportional to N. This is slightly less than an FFT, which requires $N\log_2 N$ operations.

It is easy to see that the DWT is better suited to compression problems than pattern matching problems, because the “basis” functions can be derived from the mother wavelet. Current research is focusing on developing optimal signal-dependent wavelets.

One can also intuitively see the similarity to fractals. There are several other popular time-frequency analysis methods, including the Wigner distribution:

$$W(t, f) \equiv \int x\left(t+\frac{\tau}{2}\right) x^*\left(t-\frac{\tau}{2}\right) e^{-j2\pi f\tau} d\tau$$