10/13/08

Name: GERALD GODBOLD

Problem	Points	Score
1(a)	15	
1(b)	15	
1(c)	15	
2(a)	15	
2(b)	15	
3(a)	15	
3(b)	10	
Total	100	

Notes:

(1) The exam is closed books and notes except for one double-sided sheet of notes.

(2) Please indicate clearly your answer to the problem.

(3) The details of your solutions are more important than the answers. Please explain your solutions clearly and include as many details as possible.

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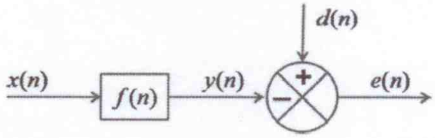
1. For the adaptive system shown to the right, assuming $f(n)$ is a linear, time-invariant moving average filter:

$$y(n) = \sum_{i=0}^{L-1} f(i)x(n-i)$$



(a) Derive the normal equations for the minimum least squares error estimate of the filter coefficients.

(b) Develop the concept of a linear prediction filter based on this model (explain how this model is modified to produce a linear prediction estimate of the filter coefficients).

(c) Derive the expression for the autocorrelation estimate of the linear prediction coefficients.

2. Modify the block diagram shown above to produce the basic LMS adaptive filter.

(a) Derive an expression for estimation of the filter coefficients using an iterative-in-time approach.

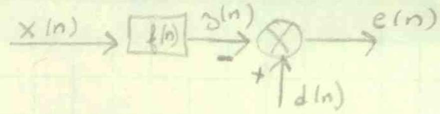
(b) Compare and contrast this to the approach in Prob. 1.

3. Suppose the input signal to the adaptive filter shown above is as follows: $x(n) = ax(n-1) + w(n)$. Assume $w(n)$ is zero-mean white Gaussian noise.

(a) Explain how successful the filter in no. 2 will be at correctly estimating the underlying parameters of this signal. Be as specific as possible and use terms such as the bias, variance, and convergence.

(b) Explain under what conditions we prefer approach no. 2 over approach no. 1. Again, be as specific as possible in your explanation, discussing issues such as, but not limited to, computational complexity.

1 a) $y(n) = \sum_{i=0}^{L-1} f(i)x(n-i)$



$e(n) = d(n) - (f_n \circledast x(n)) = d(n) - \overbrace{\sum_{i=0}^{L-1} f(i)x(n-i)}^{y(n)}$

i - SAMPLE INDEX
j - FILTER COEFF INDEX

MIN ERROR POWER $\Rightarrow J = E\{e^2(n)\} \Rightarrow \min J = \frac{\partial J}{\partial f(j)} = 0$

$\Rightarrow \frac{\partial E\{e^2(n)\}}{\partial f(j)} = 2E\{e(n)\frac{\partial e(n)}{f(j)} = 2E\{\overbrace{(d(n) - \sum_i f(i)x(n-i))}^{e(n)}\,\partial\left(\frac{\overbrace{d(n) - \sum f(i)x(n-i)}^{e(n)}}{\partial f(j)}\right)\}$

$\Rightarrow 2E\{(d(n) - \sum_i f(i)x(n-i))(-x(n-j))\}$

$= 2E\{(-x(n-j))(d(n))\} - 2E\{(-x(n-j))(\sum_i f(i)x(n-i))\}$

$= -2E\{x(n-j)d(n)\} + 2E\{\sum_i f(i)x(n-i)x(n-j)\}$

$= 2E\{\sum_i f(i)\overbrace{x(n-i)x(n-j)}^{R}\} - E\{\overbrace{d(n)x(n-j)}^{g}\}$

$= 2(\sum R(n-i)(n-j)f(i) - g(n-j)d(n))$

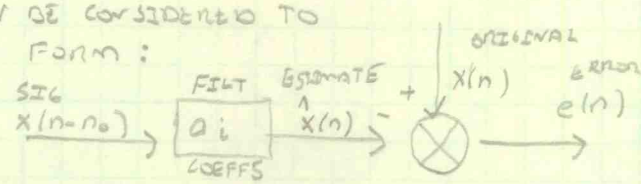
$\Rightarrow 2\sum_i R(n-i)(n-j)f(i) = 2g(n-j)d(n)$

$$\boxed{\Rightarrow \sum_i R(n-i)(n-j)f(i) = g(n-j)d(n) \qquad (1)}$$

FOR INFINITE LENGHT, STATIONARY SIGNAL, (1) REDUCES TO

$$\boxed{\Rightarrow \sum_{i=-\infty}^{\infty} r(j-i)f(i) = g(j) \qquad (2)}$$

b) FOR LP FILTER, MODEL ABOVE CAN BE CONSIDERED TO BE A p^{th} ORDER IN THE FOLLOWING FORM:



$\hat{x}(n) = \sum_{i=1}^{p} a_i x(n-n_0-i)$

$e(n) = x(n) - \hat{x}(n) = x(n) - \sum_{i=1}^{p} a_i x(n-i)$

AGAIN, $\min J = E\{e^2(n)\} = E\{(x(n) - \hat{x}(n))^2\} = E\{(x(n) - \sum_{i=1}^{p} a_i x(n-i))^2\}$

$\frac{\partial J}{\partial a_i} = 0$

$\Rightarrow E\{x^2(n)\} - a_i \sum_{i=1}^{p} E\{x(n)x(n-i)\} + E\{(\sum_{i=1}^{p} a_i x(n-i))^2\}$

$\frac{\partial J}{\partial a_i} = -2E\{x(n)x(n-i)\} + 2E\{\sum_{i=1}^{p} a_i x(n-i)\}x(n-i) = 0$

$\Rightarrow E\{\sum_{i=1}^{p} a_i x(n-i)\}x(n-i) = E\{x(n)x(n-i)\}$

COVARIANCE

$$\boxed{\Rightarrow \sum_{i=1}^{p} a_i E\{x(n-i)x(n-i)\} = E\{x(n)x(n-i)\} = c(i,j)}$$

SO, IN MATRIX FORM $\boxed{C\hat{a} = \hat{c}, \therefore \hat{a} = C^{-1}\hat{c}}$

$\boxed{1c}$ GIVEN THAT $\min J = \sum_{i=1}^{n} a_i E\{x(n-i)x(n-i)\} = E\{x(n)x(n-i)\} = c(i,j)$

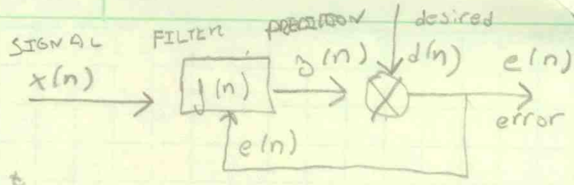
IF SIGNAL IS STATIONARY, THE COVARIANCE'S IN THE ABOVE EQUATION CAN BE CONVERTED TO SIMPLE CORRELATIONS.

THUS $C\hat{a} = \hat{c} \Rightarrow \boxed{R\hat{a} = \hat{r}, \text{ or } \hat{a} = R^{-1} r}$

SOLVE FOR FILTER COEFF'S

* R IS TOEPLITZ MATRIX, i.e. POSITIVE-SEMI DEFINATE, AND THE INVERSE WILL EXIST!
* CORRELATION METHOD WILL REQUIRE P+1 VALUES TO YEILD P COEFF'S, WHILE THE COVARIANCE METHOD REQUIRES $p(p-1)/2$,

2 LMS



DEFINE DATA: $\hat{x}_n = [x(n), x(n-1) \ldots x(n-L)]^t$

FILTER COEFFS: $\hat{h}_n = [h_n(0), h_n(1) \ldots h_n(L+1)]^t$

ERROR : $e(n) = d(n) - y(n)$ $\quad y(n) = \sum_{i=0}^{L-1} h_n(i)\, x(n-i) = \hat{h}_n^t \hat{x}_n$

OPTIMAL

PROPOSE ITTERATION: $\underbrace{\hat{h}^*_{n+1}}_{} = \underbrace{\hat{h}_n}_{\text{OLD}} - \underbrace{\alpha_n}_{\text{STEP SIZE}} \underbrace{\hat{p}_n}_{\text{DIRECTION VECTOR}}$

PROPOSE DIRECTION TO BE STEEPEST DESCENT, i.e. NORMAL TO ERROR

$\hat{p}_n = \underbrace{\tilde{D}_n}_{\text{WEIGHTS}} \nabla J_n$, $\quad \nabla J_n = \frac{\partial J}{\partial \hat{h}_n} = \left(\frac{\partial J}{\partial h(0)}, \frac{\partial J}{\partial h(1)} \ldots \right)$

SO, LET $D = I$, AND DIRECTIO $p_n = \frac{\partial E\{(e^2(n)\}}{\partial h_n} = 2(R\hat{h} - \hat{s})$

$\therefore \hat{h}_{n+1} = \hat{h}_n - \alpha(R\hat{h}_n - \hat{s})$

IN THIS CASE, R MUST BE KNOWN, SO WE CAN APPROXIMATE R BY SAYING $E\{e^2(n)\} = e^2(n)$ (INSTANTANEOUS ERROR)

BY THIS SIMPLIFICATION $\Rightarrow$ $\hat{h}_{n+1} = h_n - \frac{\alpha}{2}\nabla J_n = h_n - \frac{\alpha}{2}\, 2e(n)\frac{\partial e(n)}{\partial h(n)}$

$$\boxed{\therefore \hat{h}_{n+1} = \hat{h}_n + \alpha\, e(n)\, \hat{x}_n}$$

b) THE LMS ADAPTIVE METHOD PRODUCE A RESULT WHICH CAN BE SOLVED EFFICIENTLY IN A STEP-BY-STEP MANNER, AND THE α TERM CAN BE IMPLEMETED IN MANY WAYS TO VARY ITS CONVERGENCE PROPERTIES. THE LP METHOD REQUIRES THE R^{-1} TO BE CALCULATED, WHICH CAN BE DONE USING RECURSION INSTEAD OF INVERSE CALCULATIONS.

THE CONVERGENCE OF LMS GUARANTEED STABLE FOR $0 < \alpha < \frac{2}{L(\text{POWER OF FILT})}$.

BUT, IT IS NOT UNIFORM AND DEPENDS ON SPREAD OF THE EIGENVALUES IN R.

[3] $x(n) = a\,x(n-1) + w(n)$

AUTOREGRESSIVE, LP FILTER
ONE POLE,

$$X(z) = \frac{1}{a z^{-1}} + W(z)$$

a) THE INPUT SIGNAL HAS A CONTINUOUS BIAS a, AND UNIFORM NOISE DISTRIBUTION WITH MEAN μ_w AND CONST VARIANCE σ_w^2 OVER THE ENTIRE SPECTRUM. (AUTO REGRESSIVE MODEL)

WITH SUFFICIENT LENGTH, THE LMS FILTER WILL BE SUCCESSFUL IN SEPERATING THE NOISE COMPONENT FROM THE SIGNAL. WITH EACH ITTERATION, THE SIGNAL & NOISE WILL BE DRIVEN MORE AND MORE ORTHOGANAL. CONVERGENCE WILL BE GUARANTEED FOR $0 < \alpha < 2/L$. THE RATE OF CONVERGENCE WILL NOT BE UNIFORM, AND WILL BE A FUNCTION OF THE SPREAD OF THE EIGENVALUES IN R. IF NOISE COMPONENT $w(n)$ WAS NON-STATIONARY, THIS WOULD NOT BE THE CASE!

b) THE LMS ADAPTIVE FILTER IS PREFFERED FOR THE FOLLOWING:

1) IDEAL FOR REAL-TIME APPLICATIONS. UPDATES ARE MADE SAMPLE-BY-SAMPLE.
2) SIMPLE $O(L)$ COMPUTATION COMPLEXITY, AS OPPOSED TO $O(L^2)$ OR $O(L^3)$ WHERE INVERSES OR INVERSE APPROXIMATIONS MUST BE CALCULATED. PROPORTIONAL TO FILTER, NOT SIGNAL!
3) ADAPTION RATE CAN BE CONTROLLED AND VARIED BY HYPOTHESIZING VARIOUS METHODS OF DETERMINING α.
4) CONVERGENCE CAN BE GUARANTEED.
5) MANY MORE COMPLEX VARIATIONS EXIST, SUCH AS LEAKY LMS (i.e "FORGETTING FACTOR") $\mathfrak{J}^*_{n+1} = \gamma\,\mathfrak{J}_n + \alpha\, e(n)\,\hat{x}_n$, PILOT LMS, CLIPPED, SIGN-SIGN, ECT... LOTS OF INTERESTING VARIATIONS!