

Name: GERALD GODBOLD

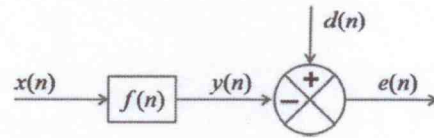
Problem	Points	Score
1(a)	15	
1(b)	15	
1(c)	15	
2(a)	15	
2(b)	15	
3(a)	15	
3(b)	10	
Total	100	

Notes:

- (1) The exam is closed books and notes except for one double-sided sheet of notes.
- (2) Please indicate clearly your answer to the problem.
- (3) The details of your solutions are more important than the answers. Please explain your solutions clearly and include as many details as possible.

1. For the adaptive system shown to the right, assuming $f(n)$ is a linear, time-invariant moving average filter:

$$y(n) = \sum_{i=0}^{L-1} f(i)x(n-i)$$



(a) Derive the normal equations for the minimum least squares error estimate of the filter coefficients.

(b) Develop the concept of a linear prediction filter based on this model (explain how this model is modified to produce a linear prediction estimate of the filter coefficients).

(c) Derive the expression for the autocorrelation estimate of the linear prediction coefficients.

2. Modify the block diagram shown above to produce the basic LMS adaptive filter.

(a) Derive an expression for estimation of the filter coefficients using an iterative-in-time approach.

(b) Compare and contrast this to the approach in Prob. 1.

3. Suppose the input signal to the adaptive filter shown above is as follows: $x(n) = ax(n-1) + w(n)$. Assume $w(n)$ is zero-mean white Gaussian noise.

(a) Explain how successful the filter in no. 2 will be at correctly estimating the underlying parameters of this signal. Be as specific as possible and use terms such as the bias, variance, and convergence.

(b) Explain under what conditions we prefer approach no. 2 over approach no. 1. Again, be as specific as possible in your explanation, discussing issues such as, but not limited to, computational complexity.

$$\boxed{1} \quad 0) \quad \hat{d}(n) = \sum_{i=0}^{L-1} j(i) x(n-i)$$

$$e(n) = d(n) - (\hat{d}(n) \oplus x(n)) = d(n) - \sum_{i=0}^{L-1} j(i) x(n-i)$$

$$\text{MIN ERROR POWER} \Rightarrow J = E\{e^2(n)\} \Rightarrow \min J = \frac{\partial J}{\partial j(i)} = 0$$

$$\Rightarrow \frac{\partial E\{e^2(n)\}}{\partial j(i)} = \frac{\partial E\{e(n) \frac{\partial e(n)}{\partial j(i)}\}}{\partial j(i)} = \frac{\partial E\{(d(n) - \sum_{i=0}^{L-1} j(i) x(n-i)) \frac{\partial}{\partial j(i)} (d(n) - \sum_{i=0}^{L-1} j(i) x(n-i))\}}{\partial j(i)}$$

$$\Rightarrow \frac{\partial E\{(d(n) - \sum_{i=0}^{L-1} j(i) x(n-i)) (-x(n-i))\}}{\partial j(i)}$$

$$= \frac{\partial E\{-x(n-i) d(n)\}}{\partial j(i)} - \frac{\partial E\{-x(n-i) (\sum_{i=0}^{L-1} j(i) x(n-i))\}}{\partial j(i)}$$

$$= -\frac{\partial E\{x(n-i) d(n)\}}{\partial j(i)} + \frac{\partial E\{\sum_{i=0}^{L-1} j(i) x(n-i) x(n-i)\}}{\partial j(i)}$$

$$= \frac{\partial E\{\sum_{i=0}^{L-1} j(i) x(n-i) x(n-i)\}}{\partial j(i)} - \frac{\partial E\{d(n) x(n-i)\}}{\partial j(i)}$$

$$= \frac{\partial \{ \sum_{i=0}^{L-1} r(n-i)(n-i) j(i) - g(n-i) d(n) \}}{\partial j(i)}$$

$$\Rightarrow \frac{\partial \{ \sum_{i=0}^{L-1} r(n-i)(n-i) j(i) \}}{\partial j(i)} = \frac{\partial \{ g(n-i) d(n) \}}{\partial j(i)}$$

$$\Rightarrow \sum_{i=0}^{L-1} r(n-i)(n-i) j(i) = g(n-i) d(n) \quad (1)$$

FOR INFINITE LENGTH, STATIONARY SIGNAL, (1) REDUCES

$$\text{TO} \Rightarrow \sum_{i=-\infty}^{\infty} r(i-j) j(i) = g(j) \quad (2)$$

b) FOR LP FILTER, MODEL ABOVE CAN BE CONSIDERED TO BE A Pth ORDER IN THE FOLLOWING FORM:

$$\hat{x}(n) = \sum_{i=1}^p a_i x(n-n_0-i)$$

$$e(n) = x(n) - \hat{x}(n) = x(n) - \sum_{i=1}^p a_i x(n-i)$$

$$\text{AGAIN, } \min J = E\{e^2(n)\} = E\{(x(n) - \hat{x}(n))^2\} = E\{(x(n) - \sum_{i=1}^p a_i x(n-i))^2\}$$

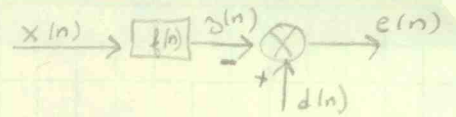
$$\Rightarrow E\{x(n)^2\} - a_i \sum_{i=1}^p E\{x(n) x(n-i)\} + E\{(\sum_{i=1}^p a_i x(n-i))^2\}$$

$$\frac{\partial J}{\partial a_i} = -2E\{x(n) x(n-i)\} + 2E\{\sum_{i=1}^p a_i x(n-i)\} x(n-i) = 0$$

$$\Rightarrow E\{\sum_{i=1}^p a_i x(n-i)\} x(n-i) = E\{x(n) x(n-i)\} \quad \text{COVARIANCE}$$

$$\Rightarrow \left[\sum_{i=1}^p a_i E\{x(n-i) x(n-i)\} \right] = E\{x(n) x(n-i)\} = c(i, j)$$

$$\text{SO, IN MATRIX FORM } \hat{c} = \hat{c}, \therefore \hat{a} = c^{-1} \hat{c}$$



i - SAMPLE INDEX
j - FILTER COEFF INDEX

$$\frac{\partial J}{\partial a_i} = 0$$

$$\boxed{1c} \text{ GIVEN THAT } \min J = \sum_{i=1}^p a_i E\{x(n-i)x(n-i)\} = E\{x(n)x(n-i)\} = C(i,j)$$

IF SIGNAL IS STATIONARY, THE COVARIANCE'S IN THE ABOVE EQUATION CAN BE CONVERTED TO SIMPLE CORRELATIONS.

$$\text{THUS } C\hat{a} = \hat{z} \Rightarrow \boxed{R\hat{a} = \hat{r}, \text{ OR } \hat{a} = R^{-1}\hat{r}}$$

↑
SOLVE FOR FILTER COEFF'S

* R IS TOEPLITZ MATRIX, I.E. POSITIVE-SEMI DEFENATE, AND THE INVERSE WILL EXIST!

* CORRELATION METHOD WILL REQUIRE $P+1$ VALUES TO YIELD P COEFF'S, WHILE THE COVARIANCE METHOD REQUIRES $P(P-1)/2$,

2 LMS

$$\text{DEFINE DATA: } \hat{x}_n = [x(n), x(n-1), \dots, x(n-L)]^T$$

$$\text{FILTER COEFFS: } \hat{j}_n = [j_n(0), j_n(1), \dots, j_n(L-1)]^T$$

$$\text{ERROR: } e(n) = d(n) - y(n) \quad y(n) = \sum_{i=0}^{L-1} j_n(i) x(n-i) = \hat{j}_n^T \hat{x}_n$$

$$\text{PROPOSE ITERATION: } \hat{j}_{n+1}^* = \hat{j}_n^{\text{old}} - \alpha_n \hat{p}_n$$

\uparrow STEP SIZE \leftarrow DIRECTION VECTOR

PROPOSE DIRECTION TO BE STEEPEST DESCENT, I.E. NORMAL TO ERROR

$$\hat{p}_n = \overset{\text{WEIGHTS}}{D}_n \nabla \overset{\text{WRIGHTS}}{J}_n, \quad \nabla J_n = \frac{\partial J}{\partial \hat{j}_n} = \left(\frac{\partial J}{\partial j_n(0)}, \frac{\partial J}{\partial j_n(1)}, \dots \right)$$

$$\text{SO, LET } D = I, \text{ AND DIRECTION } p_n = \frac{\partial E\{e^2(n)\}}{\partial \hat{j}_n} = 2(R\hat{j}_n - \hat{s})$$

$$\therefore \hat{j}_{n+1} = \hat{j}_n - \alpha (R\hat{j}_n - \hat{s})$$

IN THIS CASE, R MUST BE KNOWN, SO WE CAN APPROXIMATE R BY SAYING $E\{e^2(n)\} = e^2(n)$ (INSTANTANEOUS ERROR)

$$\text{BY THIS SIMPLIFICATION } \Rightarrow \hat{j}_{n+1} = \hat{j}_n - \frac{\alpha}{2} \nabla J_n = \hat{j}_n - \frac{\alpha}{2} \frac{\partial e(n)}{\partial \hat{j}_n} \frac{\partial e(n)}{\partial \hat{j}_n}$$

$$\boxed{\therefore \hat{j}_{n+1} = \hat{j}_n + \alpha e(n) \hat{x}_n}$$

- b) THE LMS ADAPTIVE METHOD PROVIDE A RESULT WHICH CAN BE SOLVED EFFICIENTLY IN A STEP-BY-STEP MANNER, AND THE α TERM CAN BE IMPLEMENTED IN MANY WAYS TO VARY ITS CONVERGENCE PROPERTIES. THE LP METHOD REQUIRES THE R^{-1} TO BE CALCULATED, WHICH CAN BE DONE USING RELAXATION INSTEAD OF INVERSE CALCULATIONS.

THE CONVERGENCE OF LMS GUARANTEED STABLE FOR $0 < \alpha < \frac{2}{L}$.

BUT, IT IS NOT UNIFORM AND DEPENDS ON SPREAD OF THE EIGENVALUES IN R. L (POWER OF FILTER)

