TOPOLOGICAL ASPECTS OF CHAOS

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ABSTRACT

Unraveling the complexity arising out of even simple chaotic systems requires the application of tools from several fields of mathematics and engineering. In this paper, the basic concepts underlying a topological description of chaos are reviewed. An overview of dynamical systems theory is provided using a qualitative description of differential equations through invariant manifolds of hyperbolic fixed points. The generative mechanism of chaos from homoclinic points is explained. Also, the application of topological knot theory in developing powerful topological invariants is outlined.

1. INTRODUCTION

Analysis of chaotic time series originating from low dimensional nonlinear dynamical systems has been one of the most challenging problems in the scientific history. Such signals were traditionally discarded as contaminated by a high degree of noise. Some of the greatest mathematicians and physicists have devoted their time and energy to develop mathematically sound and physically useful theories of chaos. This has lead to renewed hope and interest in analyzing the noise-like chaotic signals.

A theory for low-dimensional chaotic systems should consist of two interrelated components:

1. a qualitative encoding of the topological structure of the chaotic attractor [1,2,3] (symbolic dynamics, topological invariants, etc.), and
2. a quantitative description of the metric structure on the attractor [4,5] (Lyapunov exponents, fractal dimensions, etc.).

Topology is a kind of geometry that studies the properties of a space that are unchanged under a reversible continuous transformation [2]. The field of topology originated from Poincare's work on unraveling the complicated dynamics of the motion of objects in a three-body problem [6]. It is probably the best mathematical tool with which we can hope to find a breach through the complicated structures of chaotic systems.

In this paper, I present an intuitive and an informal description of how we can understand and classify chaotic systems, from a topological viewpoint. The rest of this report is organized as two parts. In the first part some basic results from the theory of dynamical systems are reviewed. This includes discussion on invariant manifolds of the fixed points and periodic orbits of hyperbolic systems, the relation of orbits of chaotic maps to symbolic dynamics and how chaos can originate from intersections between stable and unstable manifolds. Part 2 is primarily concerned with the study of topological invariants for classification strange attractors. This discussion includes some results from knot theory and the use of topological entropy, linking numbers and relative rotation rates as invariants and the concept of templates.

PART I: THEORY OF DYNAMICAL SYSTEMS

2. FIXED POINTS AND INVARIANT MANIFOLDS

In this and the following sections, the theory of dynamical systems arising from a qualitative description of solutions of differential equations is discussed. Specifically, the behavior of orbits near fixed points (and periodic orbits) holds the key to unraveling the complicated structure of chaotic systems. This is the viewpoint that will be advanced in the next few sections. See [2,3,5] for details.

Consider a set of ordinary differential equations of the form:

$$\dot{x} = F(x)$$

If the vector field F does not contain time explicitly, the system defined by the equation is said to be *autonomous*; otherwise it is said to be *nonautonomous*. Any nonautonomous system can be converted to a higher-dimensional autonomous system by rewriting the equation using more state variables to remove explicit dependence on the time parameter. Hence, from now on we will consider only autonomous systems.

The flow, ϕ, through x0 at t=0, is defined as the smooth function that satisfies:

$$\frac{d}{dt}\phi_t(x) = F(\phi_t(x))$$
$$\phi_0(x) = x_0$$

The position x_0 is the *initial condition* (or initial state) and an individual solution through x_0 is called a *solution curve*, *trajectory* or *integral curve*. The collection of states of a system is called the *phase space*. The existence and uniqueness of the solution is guaranteed (by the existence and uniqueness theorem) under some reasonable assumptions on the properties of the vector field.

The discrete-time analog of a flow is the map. We can go from a flow to a map using Poincare sections. A time-T map where the flow is sampled every T time units is one simple example of a Poincare map. The opposite process - making a flow out of a map, is called a suspension of map. The construction of suspensions is far from unique. In most cases, results and definitions pertaining to flows have equivalent counterparts for maps, and vice versa.

We will assume that our system is *dissipative*. For such systems, the phase space is continually shrinking into a smaller region of the phase space. This leads to the concept of invariant bounded attractors. (A set S is an *invariant* set if a flow if for any x_0 in S we have $\phi_t(x_0) \in S$ for all $t \in R$). Poincare-Bendixson theorem states that the only attractors possible in a 2-D (planar) vector field are the fixed points and limit cycles. No chaotic attractors are possible for such systems. Hence, the minimum phase space dimension for chaotic motion is three.

A *fixed point* (or equilibrium point) x_f is a constant, time-independent solution. At a fixed point the vector field F, vanishes. The question of stability of a fixed point is of vital importance in determining the trajectory of a point in its neighborhood. Two notions of stability that are of concern here are the *local stability* and the *linear stability*. A fixed point is locally stable (e.g. sink) if solutions based near x remain close it for all future times. Further if the solution actually approaches the fixed point, then the orbit is called asymptotically stable. A fixed point is unstable if it is not stable (e.g., source, saddle).

One main approach to studying the stability near a fixed point is through *linearization* (linear stability). Considering Taylor expansion of small perturbations about the fixed point, the motion near the fixed point should be governed by the linear system:

The eigenvalues of the Jacobian of the linearized tangent space determines the asymptotic stability of fixed points. Let $\{\lambda_i\}$ denote the set of eigenvalues. A fixed point is said to be *hyperbolic* if none of the real parts of $\{\lambda_i\}$ is equal to 0. Otherwise it is said to be *nonhyperbolic*. For nonhyperbolic fixed points, though the system is linearly stable, the nonlinear system may or may not be stable. We restrict our attention mostly only to hyperbolic systems because they are easier to tackle. Such systems cannot exhibit the phenomenon of bifurcations. However, all systems belonging to the real world are nonhyperbolic and reach the chaotic regime through a series of bifurcations of one kind or the other. Fortunately, most of the theory developed for hyperbolic systems work adequately well for nonhyperbolic systems too.

Now we will consider some special cases of the distribution of the eigenvalues about zero. If all the real parts of $\{\lambda_i\}$ are negative, the linear system is stable and so also is the original nonlinear system. Such fixed points are called *stable nodes* or *sinks*. If all the real parts are greater than zero then the linear system and also the actual nonlinear system are unstable. Such fixed points are called *unstable nodes* or *sources*. If for a hyperbolic fixed point, at least one real part is greater than zero and at least one real part less than zero, then it is said to be a *saddle point*. There is another type of fixed point called the *center* that that occurs only in nonhyperbolic systems. For a center, the eigenvalues are purely imaginary. It should be mentioned here that all of these could be reinterpreted for maps by considering the behavior of the modulus of eigenvalues with respect to one.

For a fixed point it makes sense to consider which set of points that asymptotically converge to it and which do not. The eigenspaces of a linear flow are invariant subspaces of the dynamical system. The dynamics on each subspace are determined by the eigenvalues of the subspace. If the original manifold is R_n then each invariant subspace is also a Euclidean manifold, which is a subset of R_n. For a linearized system, we can classify each of these invariant submanifolds according to the real parts of the eigenvalues as:

1. E^s: the (stable) subspace spanned by the eigenvectors with $\mathrm{Re}(\lambda_i) < 0$
2. E^u: the (unstable) subspace spanned by the eigenvectors with $\mathrm{Re}(\lambda_i) > 0$
3. E^c: the (center) subspace spanned by the eigenvectors with $\mathrm{Re}(\lambda_i) = 0$.

These are illustrated in Fig. 1 for a hyperbolic system (hence there is no center manifold).

But what does the invariant manifolds of the linearized system tell about the original nonlinear system? The relation comes from the *Center Manifold theorem*. This states that for the nonlinear system, there exist smooth stable and unstable manifolds, called W^s and W^u, tangent to E^s and E^u at x_f, and a center manifold W^c tangent to E^c at x_f. The manifolds W^s, W^u and W^c are invariant for the flow. The stable (W^s) and unstable (W^u) manifolds are unique. The center manifold (W^c) need not be unique. For hyperbolic systems we can decompose the neighboring

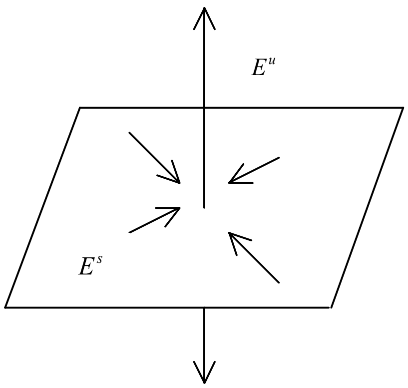

Figure 1: Local stable (E^s) and unstable (E^u) invariant manifolds of the linearized dynamics about a fixed point. Here there are two distinct stable eigenvalues – hence stable manifold is a plane. The unstable manifold is a line because there is only one unstable eigenvalue

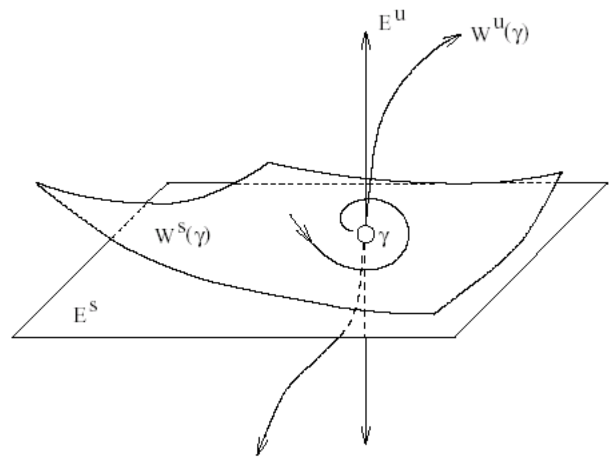

Figure 2: Local stable (W^s) and unstable (W^u) invariant manifolds of the nonlinear system about a fixed point. Here there are two stable eigenvalues – hence it is a R^2 manifold. The unstable manifold is R^1 because there is only one unstable eigenvalue. The nonlinear manifolds are tangent to the linearized manifolds (reprinted from [3]).

region into stable and unstable manifolds as is shown in Fig. 2.

Until now we focused our attention only on fixed points. The notions of invariant manifolds can also be extended to periodic orbits. A *periodic orbit* can be identified as a fixed point of an appropriate Poincare section. Hence the stability of the periodic orbit can be studied by analyzing its corresponding fixed point of the map defined by the Poincare section. In this case, characteristic exponents called Floquet multipliers play the role of eigenvalues.

3. HOMOCLINIC AND HETEROCLINIC POINTS

Informally, we can define the stable (unstable) manifold about a fixed point as those points in its neighborhood, which asymptotically reach it in the positive (negative) flow of time. In the previous section we learnt that these are tangent to the eigenvectors of x_f.

The local invariant manifolds just by themselves are not so fascinating. But these become very interesting when the stable (W^s) and unstable (W^u) invariant manifolds (of the same fixed point or of different fixed points) intersect [2,3,5]. One possible scenario occurs when the stable manifold exactly matches the unstable manifold. Such interactions are exceptional. The more common and also interesting scenario is the *transversal intersection* between the stable and unstable manifolds. If this transversal intersection point is between the stable and unstable manifolds of the same fixed point (or periodic orbit), then it is called a *homoclinic point*. If the stable and unstable manifolds are from different fixed points (or periodic orbits) then it is called a *heteroclinic point*. It should be noted that we do not consider the intersection of two stable manifolds or of two unstable manifolds as these are not possible; such an intersection would violate the theorem of uniqueness of solutions through any point. An example of a highly simplified set of homoclinic and heteroclinic points is shown in Fig. 3.

The properties of homoclinic and heteroclinic points make them highly interesting and can be use to describe the generative process of chaos. (Henceforth we speak only of homoclinic points although all of the following are true for heteroclinic points as well). A homoclinic point, by definition, lies on both the stable and unstable invariant manifold. The forward and backward map of this point should again be on both the stable and unstable manifolds, and hence should itself be a homoclinic point. This means there exists an infinity of such homoclinic points. Moreover, the stable (W^s) and unstable (W^u) manifolds must oscillate more and more wildly between the iterations of the map, $M_n(x)$ as $n \to \infty$. Also, between any two intersections there exists a dense set of intersections. Another important fact is that in any neighborhood of a homoclinic point (or its associated fixed point), there are an infinite number of periodic points of the map. But how does all this lead to chaos? This link comes from a study of symbolic dynamics.

4. SYMBOLIC DYNAMICS AND CHAOS

Consider the *sequence space* Σ_2 on the two symbols 0 and 1:

$$\Sigma_2 = \{ s = (s_0 s_1 \ldots) \mid s_j = 0 \textbf{ or } 1 \}$$

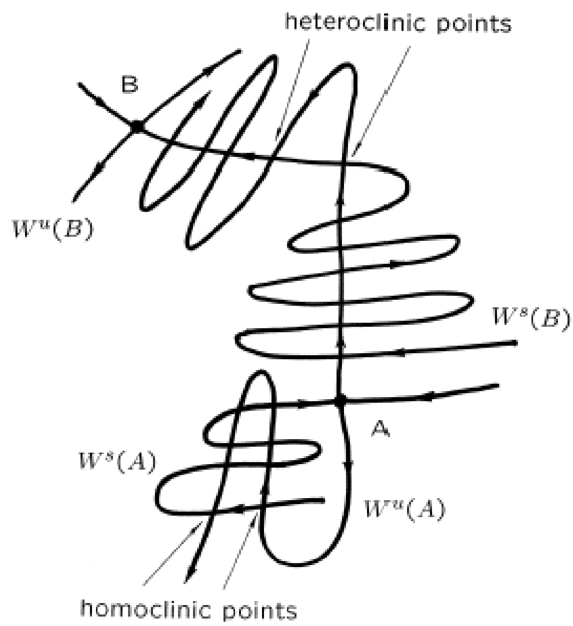

Figure 3: Homoclinic and heteroclinic points of transversal intersections between stable (W^s) and unstable (W^u) invariant manifold (reprinted from [3]).

The *shift map* on this space $\sigma\left(s_0 s_1 s_2 \ldots = \left(s_1 s_2 s_3 \ldots\right)\right.$ is defined as $\sigma\left(s_0 s_1 s_2 \ldots\right) = \left(s_1 s_2 s_3 \ldots\right)$. The shift map simply "forgets" the first entry in a sequence, and shifts all other entries one place to the left. There are several interesting properties of the shift map [2,5]. First, it is exhibits sensitivity to initial conditions. Two points sitting close to each other in the sequence space eventually separate. There exists a dense set of periodic orbits in Σ_2 and even more "number" of aperiodic orbits. (In fact, the set of periodic points can be mapped to the set of rational numbers in a real interval and the set of aperiodic orbits to the irrational numbers in the same interval). Another interesting property of Σ_2 is that there are nonperiodic orbits that come arbitrarily close to any given sequence in the space. Such orbits are called *dense* and maps that possess dense orbits are said to be *topologically transitive* (leading to the phenomena of mixing).

In our construction of the sequence space we considered only two symbols, 0 and 1. We can extend this to a more general setting containing N symbols. The dynamics on the sequence spaces of these symbols is called *symbolic dynamics*.

Another restriction that can be applied to the dynamics of symbols is by not allowing certain symbol sequences or transitions. For example, if there are three symbols 0, 1 and 2, we can allow all transitions between symbols except that from 2 to 0. Hence, we can construct a *Markov transition matri*x whose entries are either a 1 or a 0 depending on whether a particular transition is allowed or not. The shift map for this constrained sequence space is called a *subshift of finite type*.

What use is of symbolic dynamics in chaotic systems? The answer comes from the properties of hyperbolic systems with homoclinic points. It has been shown that for such systems [2,3,5], homoclinic intersections imply horseshoe type (a kind of map with a stretch and fold mechanism in one direction and squeezing in another) dynamics for some sufficiently high iterate of the map. The dynamics of the horseshoe map can be shown to have a direct correspondence with that of the shift map on symbol sequences, i.e., there is a topological conjugacy between the two dynamics. This link not only helps us in a qualitative description of chaotic systems, but we can also use symbols to code the trajectories and periodic orbits of chaotic signals to aid in their classification.

PART II: TOPOLOGICAL INVARIANTS

5. KNOTS, LINKS, BRAIDS, AND PERIODIC ORBITS

Knot theory [2,7] studies the placements of one-dimensional objects called strings in a three-dimensional space. A knot is made by taking a rope and splicing the ends together to form a closed curve. A collection of knots is called a *link*. An *oriented knot* is a closed non-intersecting loop with a sense of direction associated with it. An example of a knot and a link is shown in Fig. 4.

Another structure that is found very useful when studying knots is a *braid*. A braid is constructed between two horizontal lines with n base points. Each base point in the upper line is connected to one and only one base point of the lower line. A braid is closed by joining the lower base points to the upper base points to create links (or knots). It is proved that any oriented link can be represented as a closed braid (Alexander's theorem [2,7].

Two knots (or links) are said to be *topologically equivalent* (or *homeomorphic*) if there exists a continuous transformation carrying one knot (or link) to another. Showing that two oriented knots (or links) are equivalent is an extremely difficult problem. This can be simplified by considering the projection of knots on to a plane. This introduces crossings C, in the planar diagram each of which is assigned a sign $\sigma(c) = \pm 1$ depending on whether the crossing is right-handed (overcross) or left-handed (undercross). Reidemeister observed that two different planar diagrams of the same knot represent topologically equivalent knots under sequence of just three primary moves, now called *Reidemeister moves*. But these moves

i) A trefoil knot ii) A Hopf link

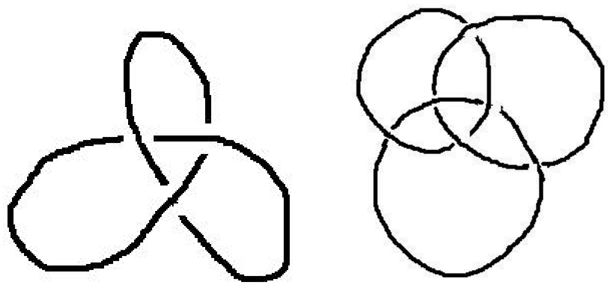

Figure 4: Example of i) a (trefoil) knot, ii) a (Hopf) link

can be applied in an infinite number of combinations, and so showing knot equivalence still remains a hard problem.

The importance of knots in a three-dimensional dynamical system arises from the fact that a periodic orbit of such a system forms an oriented closed non-intersecting curve, and is hence an oriented knot [2,3]. More importantly, the various periodic orbits form a link. In the chaotic regime, this link is extraordinarily complex, consisting of an infinite number of unstable periodic orbits (knots). This linking of the periodic orbits fixes the topological structure of a three-dimensional flow. Hence, knowledge about the organization of periodic orbits of the flow provides all the information about its strange attractor. If we can understand how to classify this organization of the periodic attractors, then we can classify strange attractors.

6. INVARIANTS OF PERIODIC ORBITS

The organization of periodic attractors (which form a link) can be classified according to certain topological invariants called knot and link invariants [1,2,3]. A *topological invariant* of a knot or a link is a quantity that does not change under continuous deformation of the strings. There exists very powerful knot invariants like the Jones polynomial and the Alexanders polynomial, but for our purposes simpler invariants will suffice.

A crude invariant that resembles the metric invariants (like Lyapunov exponents, fractal dimensions and correlation entropy [4,5]) more than the other topological invariants is the *topological entropy* [1,2]. Consider two different periodic orbits of the same period. The two periodic orbits could just be cyclic permutations of one another, in which case they are not really distinct orbits. The other case is when they are distinct orbits. For maps in one dimension, the topological entropy is defined as a measure of the growth of the number of distinct periodic cycles as a function of period, i.e., $h = \lim_{n\to\infty} \ln \frac{N_n}{n}$, where N_n is the number of distinct periodic orbits of period n. This invariant is a function of the parameter and has high positive values for chaotic regimes and low negative values in non-chaotic regimes. Positive entropy is usually taken as an indication for the amount of chaos.

The *(Gauss) linking number* [1,2,3] is a simple topological invariant defined for a link on two oriented strings A and B as the sum of the crossing numbers (sign) for each cross between A and B:

$$L(A,B) = \frac{1}{2}\sum_{C} \sigma(C)$$

An *n-component link* is an ordered collection of n disjoint knots. For an n-component link, there is a linking number associating every pair of components.

When the phase space in R^3 is a solid torus ($D^2 \times S$, where D^2 is a planar disc and S is the circle), another related invariant called the relative rotation rate can be defined [1,2,3]. For two periodic orbits A and B with respective *topological periods* (the number of distinct points a periodic orbit intersects the Poincare section) p_A and p_B, the relative rotation rate is computed as:

$$R_{ij}(A,B) = \frac{1}{p_A p_B}\sum_{n} \sigma_{i+n}\sigma_{j+n} \quad n = 1,2,\ldots, p_A p_B$$

Here i and j denote the starting points for the computation of the relative rotation rates. Hence there are $p_A p_B$ such relative rotation rates possible between A and B. Linking number is related to the relative rotation rate by:

$$L(A,B) = \sum_{i,j} R_{ij}(A,B)$$

$i = 1,2,\ldots, p_A$ and $i = 1,2,\ldots, p_B$

The relative rotation rates of all pairs of periodic orbits of a map can be placed in a matrix called intertwining matrix. This matrix is a powerful invariant for low-dimensional chaotic signals and can be used for classifying chaotic attractors. The relative rotation rates can be computed from an experimental chaotic signal and arranged into an intertwining matrix. This can be compared with the inter-twining matrices of known chaotic systems, and hence can be used for classifying strange attractors. The symbolic dynamics described in section 4, has been extremely useful in coding these periodic orbits and has facilitated the computation of the above-mentioned invariants greatly.

Mathematicians have developed a neat way of representing all the unstable periodic orbits (knots) of a chaotic attractor in a single structure called as a template (or a branched manifold or a knot holder) [1,2,3]. The key idea behind constructing a template is to recognize that for low-dimensional chaotic systems, one of the directions should be a stable one (i.e., negative Lyapunov exponents). Two points on a stable manifold shrink asymptotically to a fixed point. Thus by projecting the flow along the stable direction, we can reduce it to a planar diagram. There are

i) joining chart ii) splitting chart

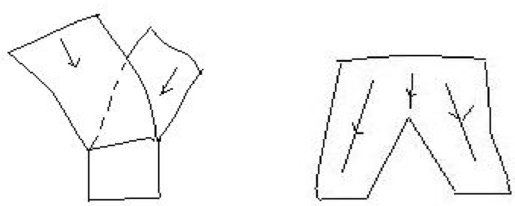

Figure 5: Template building charts i) joining chart and ii) splitting chart

two components to constructing a template: a joining chart (to identify the squeezing process) with a one-dimensional singularity called the branch line and a splitting and a splitting chart with a zero-dimensional singularity called a splitting point (that identifies initial conditions that flow into a fixed point). These are shown in Fig. 5. An example of such a template constructed from the Lorentz strange attractor is shown if Fig. 6.

The template provides complete information regarding the organization of all possible periodic orbits on the attractor in a compact form. Every attractor structure can simply be represented by a template that describes the periodic orbits and the stretching and squeezing mechanisms that forms the attractor. It is hoped that we can list out all the possible templates like the periodic table of chemical elements and that this would lead to a complete description of chaos arising in low (<4) dimensional systems.

7. CONCLUSIONS AND FUTURE WORK

In this report, an overview of how to understand chaotic systems was presented from a topological viewpoint. An introduction to a qualitative description of solutions of differential equations of dynamical systems was provided. The role played by homoclinic and heteroclinic points of hyperbolic systems in describing chaotic systems was explained through symbolic dynamics. Applications of results from knot theory in finding topological invariants from periodic orbits were also described.

This description of topological chaos provided here was by no means exhaustive. The concept of templates needs to be better understood. New approaches should be sought out to make this study more practical. For instance, the use of the knot invariants would be meaningless if the attractor does not confine itself to a subspace of three dimensions. Also, a more complete understanding of nonhyperbolic systems is needed to understand the phenomenon of bifurcations.

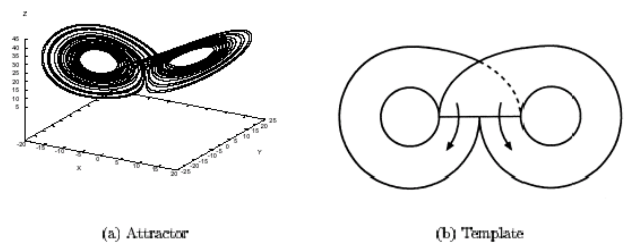

Figure 6: Lorentz attractor and its template (reprinted from from [3]).

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