Nonlinear Dynamics Modeling using Piecewise Affine Systems

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Abstract

Recent research has found that piecewise-affine systems can be used to accurately model attractor topology of certain nonlinear systems. These models are represented by a system of linear differential equations and can accurately predict the nonlinear dynamics of an attractor. This paper analyzes some of the recent research on these types of models and techniques for estimating the parameters for these models. This paper also includes an example of how this method can be used to synthesize the Rössler and Lorenz systems.

1. Introduction

The modeling of nonlinear dynamical systems is a key topic in research literature today. Most mathematical and statistical modeling techniques under study focus on accurately modeling the key features of the chaotic attractors of dynamical systems. There are two different approaches to the modeling problem: direct and inverse. The direct approach assumes that we know the equations, state variables and other significant parameters of the system of interest. In this case, it is relatively simple to build a model to approximate the system. The inverse method assumes that we only have observable measurements generated by the system, and the system itself is an unknown. In this case, the objective is to find the simplest approximate model that captures the most significant features of the trajectories derived from these observations. In most real-word situations where we need to model a system, we must take the latter approach since we do not usually have a recipe for the generation of the system's observable time series measurements. These models do have drawbacks, one being that their terms do not usually have a meaningful physical interpretation. However, they can be successful in reproducing the observable data in a statistical sense. In other words, the model can be used to accurately synthesize a time series with the same statistical properties as the observed time series [1].

This paper will review the research being done on using piecewise-affine (PWA) systems to model nonlinear dynamic systems. Research on the use PWA systems for system modeling has been going on since the 1970s where they were used to efficiently model and simulate nonlinear circuit components [2]. More recent research involves the construction of PWA models from time-series data for synthesis and prediction of nonlinear dynamic systems, including statistical approaches to estimating model parameters [3].

The first section of this paper will provide a detailed definition of PWA systems and how they can be used to model nonlinear dynamics. The second section will discuss the significance of attractor topology and why it is important to the construction of a PWA model. The final section will cite work done by [4] and show how this modeling technique can be used to accurately synthesize the Rössler and Lorenz systems.

2. Piecewise Affine Models

A piecewise-affine (PWA) system is a type of hybrid system in which the continuous dynamics within each discrete mode are affine, and the mode switching is limited to very specific regions of the subspace and is known *a priori.* [5]. The term *affine* refers to a geometry of vectors not involving any notions of length or angle. Unlike normal vector space, affine space has no knowledge of the origin, and the idea used to be referred to as the theory of *free vectors*. In other words, in affine space, it is possible to subtract points to get another point, or add a vector to a point to get another point. However, since there is no origin, it is not possible to add two points and arrive at a new point [6].

Many nonlinear dynamical systems can be characterized by regimes of different types of behavior. This is usually true for systems in a limited range of operation; modeling processes on a large domain is usually quite difficult. However, within small operation ranges, local modeling is usually simpler since there are fewer phenomena and hence, fewer parameters to deal with. This type of modeling is called operating regime-based modeling. There are two primary research interests in this area. One is the identification of linear systems to represent the different operating regimes. The other is the identification of mechanism to accurately and automatically control the switching between the regimes [3]. Recent research has shown that it is possible to form a model of this type for a nonlinear system using a PWA system [1]. This is accomplished by defining a system of linear functions to describe the different operation regimes, and using a piecewise linear function to control the switching between the regimes. This model can then be used to predict the behavior of or synthesize the original signal. Of course, in order to use this model for applications, we need to be able to prove that it is a good model, e.g., that it is computable, stable and reproducible. To ensure stability, some techniques have involved Lyapunov-based approaches to estimate models [7][8].

3. Significance of Attractor Topology

The PWA model that we estimate for a dynamical system depends heavily on the topology of the system's attractor. When these topological characteristics are as simple as points or curves formulating a PWA system to model the attractor is relatively simple. However, *strange attractors* present much more of a modeling challenge. An attractor is described as strange if it as a non-integer dimension or if the attractors dynamics are chaotic. The Rössler and Lorenz attractors (Figures 1 and 2) are examples of strange attractors and are discussed in more detail later [7].

There has been some research on the use of fuzzy logic models as local predictors, but these models tend to subdivide the phase space into a large number of neighbors. New research has shown, however, that attractors can be described using bounding tori and that the majority of the attractor is organized around a number of fixed points surrounded by the attractor's flow trajectories. It can also be shown that the attractor can be divided into a number of domains, each associated with one of the fixed points. These types of fixed points are known as focal points. The remaining fixed points are known as saddle points and link the different flow domains together [4].

To form a PWA model for such an attractor, an separate affine subsystem must be estimated for each of the fixed points. Likewise, based on the saddle points, a piecewise switching surface is estimated to control the switching of the system between the different domains. Most of the research emphasis is focused on the choice of the switching mechanism [1][4].

To formulate the switching mechanism, we need to determine several things about the attractor's topology. As an example, we will be referring to the Rössler and Lorenz attractors. The first thing we need to compute the first return map to the Poincare section. The first-return map can help analyze the mixing caused by the "twist" in the attractor by plotting the position of the trajectory as it crosses the Poincare section against the position as it

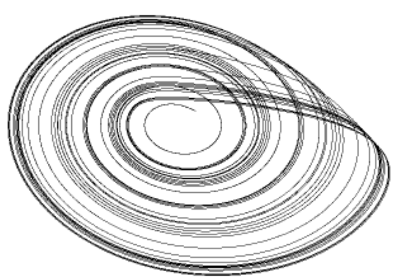

Figure 1. Rössler Attractor

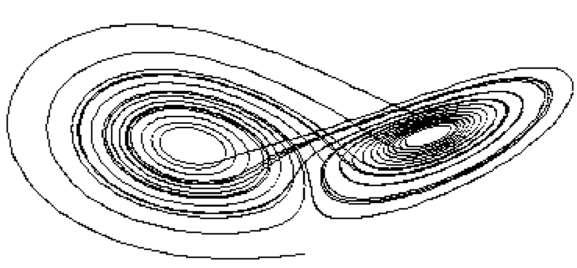

Figure 2. Lorenz Attractor

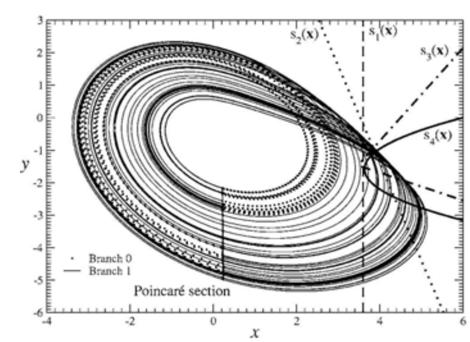

Figure 3. Poincare Section of Rössler Attractor

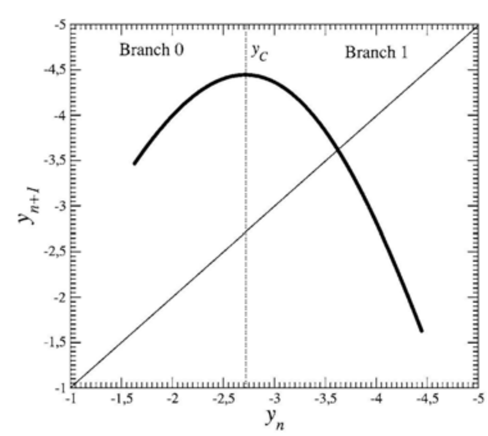

Figure 4. First-Return Map

crosses the Poincare section the next time. This is illustrated in Figures 3 and 4 [4]. More specifically, we are using the *y*-*z* plane for the Poincare section, and recording *y* each time the trajectory crosses this plane. For Rössler, this is formally defined as:

$$P \equiv \{(y_n, z_n) \in \mathbb{R}^2 | x_n = 0, x_n < 0\}$$

Notice that the first-return map is made up of two branches, each separated by a single critical point at the maximum. From these branches, we can split the phase portrait into two partitions. We can associate each side of this branch with a symbol, in this case, 0 and 1. The symbol 0 will correspond to increasing branch, and the symbol 1 to the decreasing branch. Now, trajectories can be encoded using strings of these symbols. For example, a period-*n* orbit would be represented by a symbol string of length *n*. Each element of the symbol string would correspond to the branch associated with the particular orbit. All of this information can be used to define a template for the attractor flow, which is a standard way to predict topological invariants of the attractor. The template can synthesize all of the topological properties of the attractor and the organization of the periodic orbits within it. It must also remain valid for the range of parameters where the number of branches of the first-return maps does not change [4].

Let us now define the term *linking number*. The linking number is one of the useful topological invariants predicted by the template. The attractor's periodic orbits can be thought of as knots, and the template can be viewed as the knot holder. The linking number is an indication of the number of times certain knots are crossing each other, not counting the crossing of a knot with itself. The definition of a linking number is:

$$lk(\alpha, \beta) = \frac{1}{2} \sum_{p \in \sigma} \epsilon(p)$$

The terms α and β refer to the two knots in question, and the Є(*p*) is the manner in which the two knots are crossing. In the case of a negative crossing, knot α crosses underneath β. Likewise, a positive crossing means that knot α crosses above β. In the case of a negative crossing, Є is -1. For positive crossings, Є is +1 [4].

The Lorenz system has two important topological mechanisms that are worth noting: folding and tearing. In Figure 6, **p-** and **p+** are the fixed focal points and **p0** is the saddle point. Tearing is the most prominent feature in the Lorenz system, and is induced by the saddle points mentioned earlier. Tearing is responsible for splitting the attractor flow into the two different "loops". Tearing is easy to detect in a first return map. It shows up as a non-differentiable point that shows up as a cusp. Folding, as we have seen earlier with the Rössler system, appears as an increasing branch and decreasing branch separated by a differentiable maximum (Figure 4) [4].

All of these topological observations are important to the design of a model for one of these systems. The next section will show how to use the information discussed so far to form a PWA model for the Rössler and Lorenz systems.

4. Models for Rössler and Lorenz Systems

We will now show how PWA models can be derived for the Lorenz and Rössler systems. This work was done

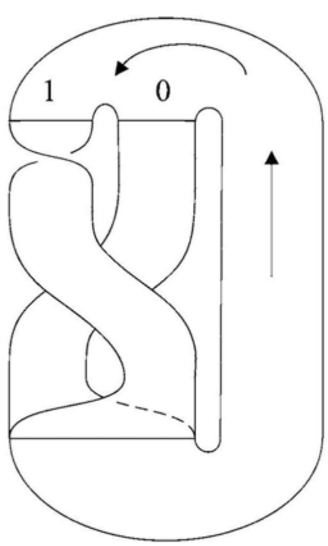

Figure 5. Rössler Template

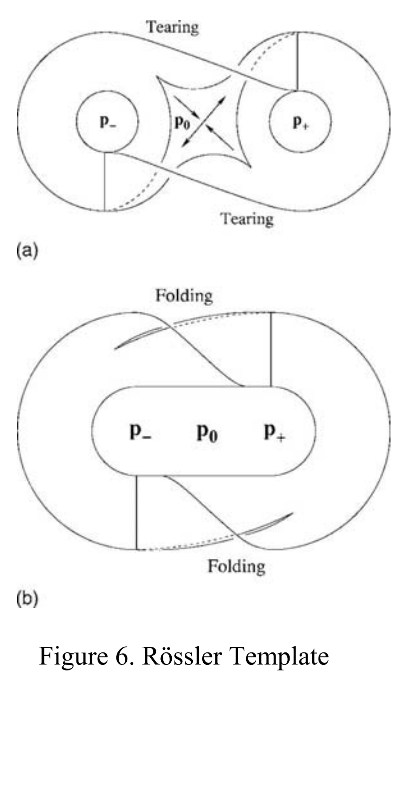

Figure 6. Rössler Template

by Amaral, Letellier, and Aguirre in [4]. The Rössler system is actually a simplified version of the Lorenz system, so the derivation of models for both of these systems will use a similar procedure. The Rössler system is defined by:

$$\begin{aligned}\dot{x} &= -y - z\\ \dot{y} &= x + ay\\ \dot{z} &= b + (x - c)z\end{aligned}$$

and has fixed points at:

$$\mathbf{p}_{\pm} = \left| \begin{array}{l} x_{\pm} = -ay_{\pm} \\ y_{\pm} = \dfrac{c \pm \sqrt{c^2 - 4ab}}{2a} \\ z_{\pm} = -y_{\pm}. \end{array} \right.$$

The points **p+** and **p-** correspond to fixed focal points. When the trajectory for the Rössler system is plotted, it is difficult to see both fixed points. The fixed point **p-** is obvious (Figure 1) and is the point around which the flow revolves. The second fixed point is responsible for influencing the folding behavior. In the first-return map (Figure 4), the increasing mode corresponds to the influence of **p-** and the decreasing mode corresponds to the influence of the point **p-**.

The Lorenz system is defined by:

$$\begin{aligned}\dot{x} &= -sx + sy\\ \dot{y} &= rx - y - xz\\ \dot{z} &= -bz + xy\end{aligned}$$

and has fixed points at:

$$\mathbf{p}_{+} = \left| \begin{array}{l} x_{+} = \sqrt{b(r-1)} \\ y_{+} = \sqrt{b(r-1)} \\ z_{+} = r - 1 \end{array} \right. \quad \mathbf{p}_{0} = \left| \begin{array}{l} x_{0} = 0 \\ y_{0} = 0 \\ z_{0} = 0 \end{array} \right. \quad \mathbf{p}_{-} = \left| \begin{array}{l} x_{-} = -\sqrt{b(r-1)} \\ y_{-} = -\sqrt{b(r-1)} \\ z_{-} = r - 1 \end{array} \right.$$

The role of the fixed points for Lorenz is different than that for Rössler. The two fixed-focal points are **p+** and **p-** and the point **p0** is a fixed saddle point. It has been shown that the only way two fixed-focal points can be surrounded by flow is if they are connected by a saddle point. The saddle point is also the primary contributing factor to the decision of a switching surface. The primary goal will be to assign an affine subsystem to each of the fixed points.

There are several steps to the procedure of building a PWA model. The first step is to determine how many affine subsystems will comprise the model. As mentioned earlier, this is determined by the number of fixed focus points. In the case of Rössler and Lorenz, these points are **p+** and **p-**. Thus, there will be two affine subsystems. The next step is to determine the switching law. The overall structure of a PWA model can be described as:

$$\dot{\mathbf{x}} = \sum_{i=1}^{m} f_i[s(\mathbf{x})]A_i(\mathbf{x} - \mathbf{p}_i)$$

where **x** is the state vector, *m* is the number of affine subsystems and **p** is the fixed points associated with the subsystem. The function s(**x**) is the switching law that determines which subsystem is active, and the matrix A defines the linear dynamics of the corresponding subsystem. The function f[.] is the Boolean function:

$$f_i[s(\mathbf{x})] = \begin{cases} 1 & \text{if } s(\mathbf{x}) \geqslant 0 \\ 0 & \text{if } s(\mathbf{x}) < 0 \end{cases}$$

From the above definitions we see that there are four things that need to be determined.

1. The fixed points **p**
2. The number of subsystems (based on the number of fixed points **p**)
3. The dynamics of each subsystem
4. The switching surface s(**x**)

For Rössler, we know that there are two fixed points and that we will have a subsystem for each. To determine the matrices A, we find the Jacobian matrices of the Rössler system evaluated at the two fixed points. Therefore the PWA model for the Rössler system is for a given set of parameters (a,b,c) is:

$$\begin{aligned}\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{bmatrix} &= f_1[s(\mathbf{x})]\begin{bmatrix}0 & -1 & -1\\ 1 & 0.398 & 0\\ z_- & 0 & (x_- - 4)\end{bmatrix}\begin{bmatrix}x - x_-\\ y - y_-\\ z - z_-\end{bmatrix}\\ &+ f_2[s(\mathbf{x})]\begin{bmatrix}0 & -1 & -1\\ 1 & 0.398 & 0\\ z_+ & 0 & (x_+ - 4)\end{bmatrix}\begin{bmatrix}x - x_+\\ y - y_+\\ z - z_+\end{bmatrix}\end{aligned}$$

Now, we need to find the switching surfaces. The specific details for estimating switching surfaces can be found in [3]. Basically, it is determined by all of the topology properties discussed previously, but the surface must lie beyond the threshold at which the nonlinearity is active. The surface must also take into consideration the findings in the first-return map to the Poincare section. In this case, the folding influenced by **p+**. For Rössler, the optimal switching surface was found to be:

$$s(\mathbf{x}) = \begin{cases} \{\mathbf{x} \in \mathbb{R}^3 | -x + 0.7y + 4.5 = 0\}, & \text{if } y \geqslant -1.482 \\ \{\mathbf{x} \in \mathbb{R}^3 | x + 2.35y - 0.02 = 0\}, & \text{if } y < -1.482 \end{cases}$$

It is now possible to use this model to synthesize the Rössler system. Figure 7 shows the synthesized attractor and resulting first-return map. Although the synthesized attractor is not identical to the actual attractor given the same parameters, the topological invariants are the same.

Recall that the fixed points for the Lorenz system are much different than those for the Rössler system. The Lorenz system has three fixed points, two being focal points and the remaining one a saddle point. The saddle results in a tearing attribute between the two attractor flows around the focal points. For the model we build in this example, we use the parameters (s,r,b)=(10,28,8/3) and calculate the fixed points to be:

$$\mathbf{p}_+ = \begin{vmatrix} 3\sqrt{8} \\ 3\sqrt{8} \\ 27 \end{vmatrix} \quad \mathbf{p}_0 = \begin{vmatrix} 0 \\ 0 \\ 0 \end{vmatrix} \quad \mathbf{p}_- = \begin{vmatrix} -3\sqrt{8} \\ -3\sqrt{8} \\ 27 \end{vmatrix}$$

The ease at which these models are estimated is due to the fact that we know the fixed points of the system. In cases where the fixed points are not known, they will have to be estimated from the attractor, and the model may not be as topologically equivalent to the original attractor.

Since we again have two focal fixed points, we will have two affine subsystems. The saddle point will contribute to the switching surface. To determine the dynamics matrices A, the Jacobian matrices are calculated at each of the focal fixed points, **p-** and **p+**. Now, we can define the PWA system for Lorenz similar to the way we did for Rössler.

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = f_1[s(\mathbf{x})] \begin{bmatrix} -10 & 10 & 0 \\ 1 & -1 & -3\sqrt{8} \\ 3\sqrt{8} & 3\sqrt{8} & -\frac{8}{3} \end{bmatrix} \begin{bmatrix} x - 3\sqrt{8} \\ y - 3\sqrt{8} \\ z - 27 \end{bmatrix} + f_2[s(\mathbf{x})]$$
$$\times \begin{bmatrix} -10 & 10 & 0 \\ 1 & -1 & +3\sqrt{8} \\ -3\sqrt{8} & -3\sqrt{8} & -\frac{8}{3} \end{bmatrix} \begin{bmatrix} x + 3\sqrt{8} \\ y + 3\sqrt{8} \\ z - 27 \end{bmatrix}$$

Notice that the affine subsystems are simply transformations of each other based on the rotation symmetry of the Lorenz system. The switching surface is:

$$s(\mathbf{x}, \theta) = \{(\mathbf{x}, \theta) \in \mathbb{R}^3 \times [0 \;\; \pi] | x - (1/\tan\,\theta) y = 0\}$$

where the $\boldsymbol{\theta}$ parameter allows the switching surface to be rotated around the z axis. By varying $\boldsymbol{\theta}$, the population of unstable orbits can be adjusted. The plots in Figure 8 show the synthesized Lorenz attractor and its corresponding first-return map to the Poincare section. For this example, a value of 1.346 was used for $\boldsymbol{\theta}$. Notice

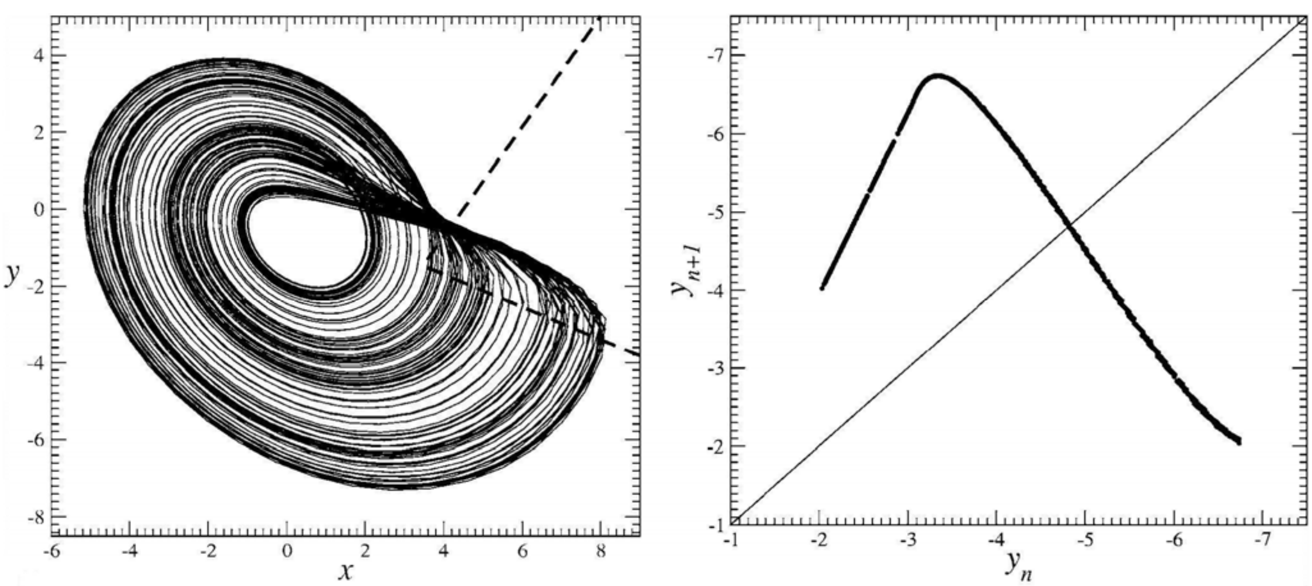

Figure 7. Synthesized Rössler System

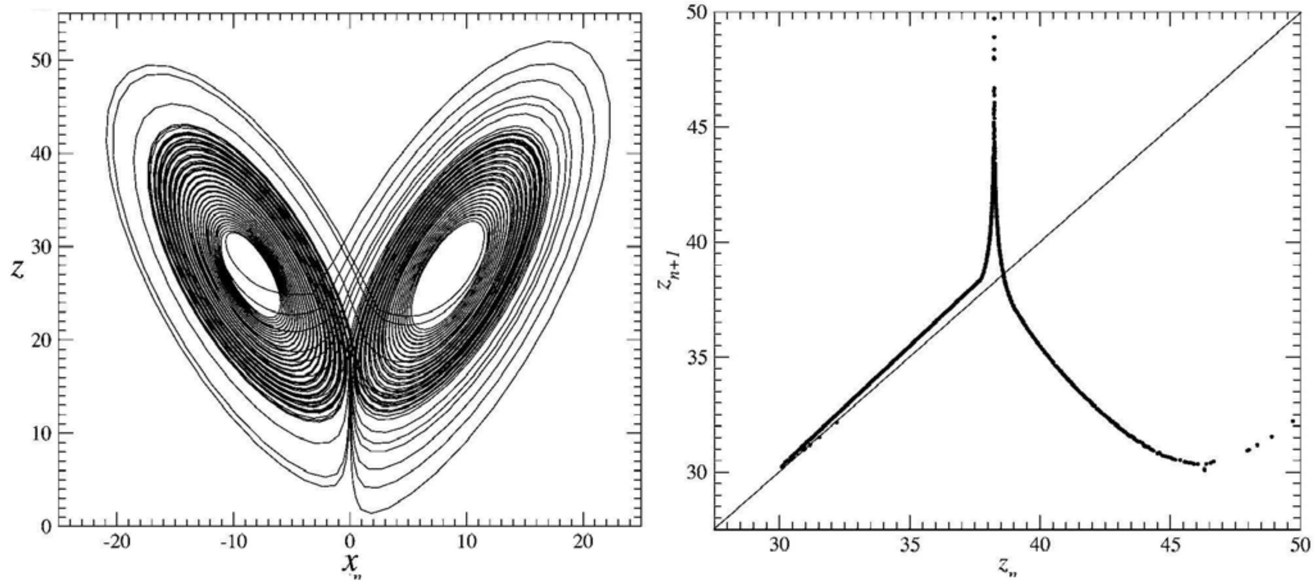

Figure 8. Synthesized Lorenz System

the non-differentiable "cusp" in the first-return map. As mentioned before, this corresponds to the tearing mechanism between the two flows in the attractor. Again, the topology characteristics are the same in the synthesized attractor as they are in the original attractor [4].

5. Summary

This paper has briefly explained how the PWA systems can be used as models for nonlinear dynamical systems. There is obviously a great deal more research needed in this area. The work in [4] has shown that PWA models can be used to synthesize and predict the attractor topology of nonlinear dynamic systems with known parameters and topology, but the building of models from observed data still presents a challenge.

6. References

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