Physical Modeling of a String Instrument

Scott Wilson, C. R. Jones, Raja Salem, Mary E. Weber

The Digital Signal Processing Group
Department of Electrical and Computer Engineering
Mississippi State University
Mississippi State, Mississippi 39762
{cjones, wilson}@ee.msstate.edu, {seelam, weber}@isip.msstate.edu

ABSTRACT

Synthesis based on physical modeling has recently become a topic of discussion among musicians looking for more a more expressive output. It has also become a topic among developers because of recent leaps in DSP technology. The main problem with physical modeling is the amount of computation required to produce output. This project explores what is necessary to produce an accurate representation of a stringed instrument - in this case, the violin, and how to produce real time output from this seemingly complex system.

Live data will be captured and analyzed to verify the fidelity of the model. A synthesis engine with a GUI front end will be used to demonstrate control over the physical model and real time audio.

1. INTRODUCTION

Physical Models of musical instruments can overcome some of the limitations of today's commercially available synthesizers. By designing a virtual model of a physical system, one has control over any aspect of that system. In the case of a musical instrument, one can control the parameters of the instrument itself, as well as control how that instrument is played. Modern synthesizers are limited primarily by the static sample of the instrument residing in memory. Any control over how this instrument is played or the physical characteristics of the instrument must be emulated. Of course, nothing comes for free. The trade-off for more low level control is higher mathematical complexity. The scope of this project is to analyze the mathematical characteristics of a violin, develop a physical model based on the wave equation, produce software that will run in real time, and compare this to that of a real instrument.

There are particular features of every instrument which causes it to produce its own characteristic sound. In order to capture those qualities in the physical model, we must first understand both quantitatively and qualitatively. This understanding can then be translated into a software model.

It is necessary to analyze the effects of the various components of the system. What effect does the bridge have? The resonating cavity? The bow? Once we have isolated the effects of each component, we can model each individually, gradually increasing the complexity of our software. We must also explore the effects of performance techniques over these components. We will then have a set of functional blocks and controllers for the effect of the element on the system. This, in effect, allows the player of the virtual instrument to have the same expressive control as the player of the real violin.

Determining the effects of each component involves solving the wave equation for the given conditions. The simplest case involves a string stretched between two rigid supports. Including the bridge in the system involves modeling a non-rigid support. Including the resonant body of the violin in the model involves knowing how it behaves as well. The results of the mathematical model can be directly compared to data from a real instrument.

Though the mathematics of the system can be described using the partial differential equations of the wave equation, the system of functional blocks and control inputs lends itself to tools available from digital signal processing. A digital waveguide based on the wave equation solution will be developed using filters, delay lines, and amplifiers. Using these simple components, it will be possible to work in real time. It will be necessary to use this method in the implementation of the synthesis software. On the other hand, the initial testing does not require real time output, but mathematical accuracy. The theory will be primarily developed using the numerical solution to the wave equation.

The ultimate objective for the outcome of this effort will be a synthesis engine with an interface that allows real time access to key parameters of the physical system. The key parameters are those determined to be associated with real performance techniques such as bow pressure, bow location, and fingering technique. In order to show that these objectives can be accomplished, we are demonstrating a selected subset of possible models. This subset includes the essential components of a violin model including the string and bridge for a plucked stimulus.

2. EXPERIMENTAL DESIGN

I. Real Violins

What distinguishes a note on a particular instrument from the same note on another instrument? The answer to this question can, hopefully, be determined from the analysis of the “real” data. By this we mean, the real data is data which uniquely determines the physical parameters of an instrument. We will consider the harmonic content to be a unique signature of an instrument and thus, the real data associated with that physical system. Because of the transient nature of the plucked string, the harmonic content will be analyzed as a function of time in order to get an accurate picture of the behavior of the string.

The real data for this project is the recorded output taken from an electric violin’s piezoelectric transducer. The data was recorded directly from the violin into a Panasonic SV-3700 DAT using a sampling frequency of 48kHz. The output of this instrument will be used as the model for a plucked string supported by one rigid support (the nut) and one non-rigid support (the bridge). Because the model instrument has no resonating body, the data recorded from it neglects any effect of this component.

This raw data is necessary to determine the harmonic content versus time of the physical system. It would be appropriate, but beyond the scope of this project, to devise a method of directly determining the displacement, velocity, and acceleration of the various components of the system. However, for our purposes; we shall have to be satisfied with the output of the above piezoelectric transducer. It is known that a piezoelectric transducer such as that found in professional musical equipment does not produce output directly related to either force or displacement, but rather a combination of the two.

Furthermore, to determine the effects of the bridge, the resonating cavity must be divorced from the system. This is conveniently the case with the electric instruments as they have no resonating cavity. Thus the real data, which we have, will be considered the signature of a plucked string with one rigid and one non-rigid endpoint. The real data will be referred to by the power spectral density (PSD) versus time. By this we mean, the power spectral density averaged over a window that is 4096 samples in length. Each sample is spaced 1/48000 seconds apart which is one of the standard sampling frequencies for digital audio. We chose the windows of 4096 samples to be overlapped by half a window width in order to maintain continuity.

II. Why Power Spectral Density?

Why is the PSD versus time significant? For a system comprised of a thin flexible string with fixed endpoints, the only frequencies that can be supported are a fundamental whose wavelength is twice the length of the string and all of its harmonics. This is because the endpoints, being fixed, determine that the waveforms pass through zero at said endpoints. Thus, the solution for the motion of the string can incorporate only those frequencies which are an integral multiple of the fundamental. For instance, if there was a waveform on the string whose frequency was not an integral multiple of the fundamental then the amplitude of the waveform at the endpoints would not be zero--for all time--and thus would not be a solution.

The fundamental frequency of the above thin flexible string with fixed endpoints is a function, only; of the string length, the string tension, and the string mass density. Given a power spectral density for a general system comprised of a thin flexible string with fixed endpoints and known tension and mass density (here after referred to as the “general system”), we should therefore be able to uniquely determine the length of the string. Or given any two of the three independent variables, we can uniquely determine the third. Since string length, tension, and mass density are easily determined by physical measurements, we can readily determine the PSD. Therefore, any deviations in the PSD will be due to the degree to which (ignoring string inhomogenities) the endpoints do not remain fixed.

The degree to which the endpoints of the above system move are uniquely determined by the physical parameters (i.e. mass, stiffness, damping, and moment-arms) of the specific instrument where the motion of the string is the forcing function. Thus in a system which is general with the exception that one endpoint moves (i.e. the bridge), the PSD should uniquely reflect the physical parameters of that endpoint.

III. The method of collection

In order to investigate the phenomena associated with the general system and the effects of movable endpoints, we need to record data indicative of the displacements, velocities, and accelerations of said endpoint. But, we were limited to the data obtained from the forces applied to the piezoelectric devices at each foot of the bridge of the electric violin.

These measurements due to the forces applied by the string are indicative of the forces which would have been applied to the resonating cavity had there been one. But, these measurements were made on an instrument with no resonating cavity and therefore the effects due to the bridge were isolated from the body of the instrument. Therefore the PSD of the recorded waveforms (frequency domain output) divided by PSD of the general system (frequency domain forcing function) will be the system transfer function (ignoring the contamination by the piezoelectric devices) of the bridge.

IV. The limitations of the data found

As stated earlier, piezoelectric devices do not produce an output relational to either force or displacement, but instead, a combination of the two. This is our primary concern as to how our result may differ from the real data. We are limited to this however, because any device capable of measuring such exact data is well above our means.

The problem stemming from the piezoelectric problem is that we are attempting to find the transfer function for the bridge isolated from the instrument. However, what we have is the transfer function of two cascaded systems. One system is the bridge and the other system is the piezoelectric device. In order to determine the transfer function of the bridge, we would have to know the transfer function of the piezoelectric device. Unfortunately, measurement of the transfer function of the piezoelectric device is outside the scope of this project. In fact, determination of the transfer function of piezoelectric device would require the destruction of the violin.

V. The analysis techniques

Basically, the real data was used in this project to analyze the data we produced using the digital-waveguide model and the numerical solution model. The characteristics of the produced data were compared with those of the real data and the parameters of our model were changed accordingly.

The analysis of the data was done strictly in Matlab. Matlab provides an easy to use interface, and a minimum amount of programming was required to produce very meaningful output. The following code segments and commands are Matlab commands.

The comparison of the data was done by plotting the attenuation of the power-spectral density with frequency and time. A fast-Fourier transform of the real was taken by windowing a position of the data.

```
S = fft(w(1:a))
```

Here w is the data file and a is the size of the window. Then the power-spectral density was calculated by using

```
R = 20log(S * conj(S)) / a
```

This data was then normalized in order to give a clearer picture of the attenuation characteristics. This allows us

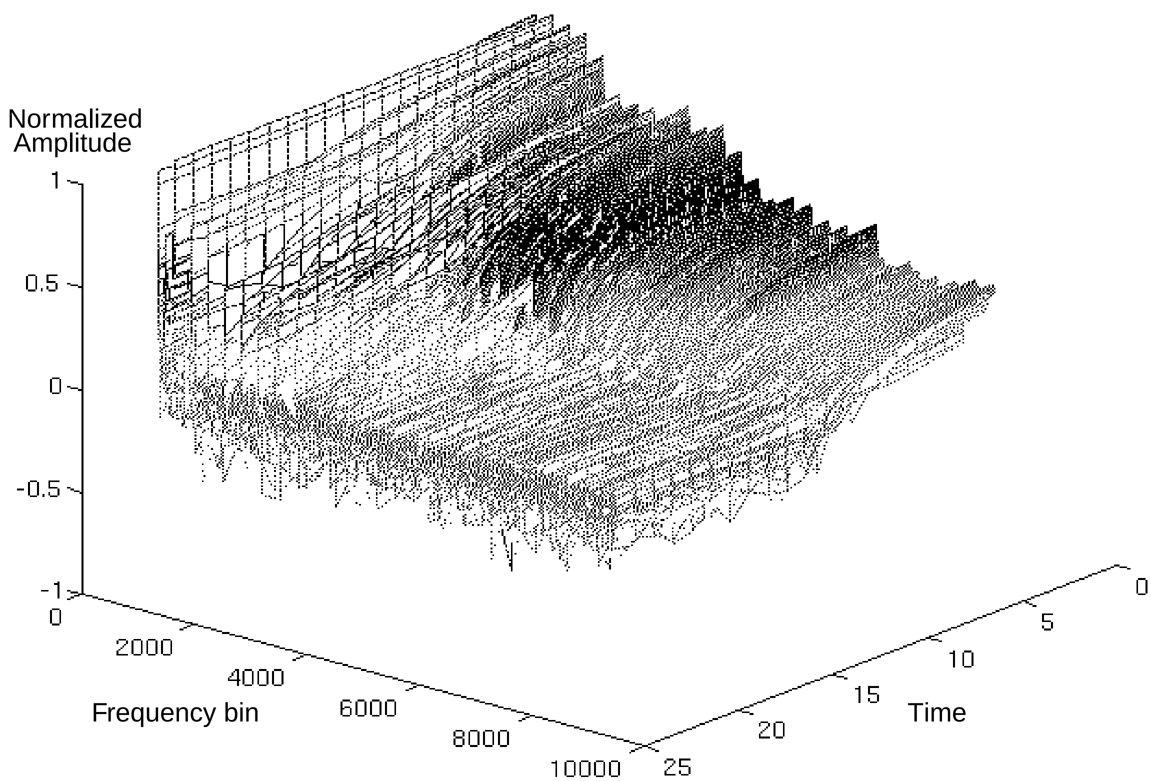

Figure 1: Harmonic spectrum of plucked violin string as function of time

to see the decay of the harmonics in relation to the fundamental, as opposed to the decay of all partials.

The subsequent calculations involved moving a window by a pre-defined position the size of the window and repeating the above calculations for these new windows. This would, essentially, result in overlapping the present window to a certain extent of the previous window enabling us to obtain a better attenuation plot. Initially, a rectangular window was used, but was later replaced by a hamming window to obtain a better plot.

VI. The results of the analysis of the data

Plots were produced for the various types of strings of the guitar and violin and the corresponding data produced by the digital-waveguide model and the data produced by the numerical solution model.

The determination of the transfer function for the real system will not be attempted. What will be undertaken as analysis is to determine, in a qualitative manner, how closely the transfer function of some “other” system (i.e. the modelled system), whose parameters we have under control, can be made to match that of the real system. If for the same input with our model, we can cause the same output as the real system; we have effectively duplicated the transfer function of the real system. Unfortunately, we do not have the luxury to make extensive measurements of real system parameters. We therefore have to take trial and error approach to determining the parameters of our model.

Analysis of the displacement of the A string stimulated by a pluck shows that the section of string attached to the bridge approximates a rectangular waveform, as shown in figure 2. The point of pluck determines the symmetry of the waveform. If the pluck is near the bridge, the waveform has an extended positive (assuming the pluck, away from the body of the instrument, is in the positive direction) portion followed by a narrow negative portion with areas under each waveform being equal so that there is no DC offset. On the other hand if the pluck is in the center of the string, the positive and negative portions of the waveform are symmetric.

The rectangular displacement waveform gives rise to rectangular force waveform. This waveform is filtered by the bridge and subsequently applied to the body of the instrument. The upper plate of an acoustic instrument undergoes displacements due to the applied forces and the velocities associated with the displacements are coupled to the air and thus radiated. For our real system, which is electric, the forces transmitted through the bridge are applied to piezoelectric devices. These devices are rigidly attached to the body of the instrument which is also rigid. Therefore, the velocity of the feet of the bridge is zero. This in effect isolates the bridge from the body of the instrument which is essential for modeling the string-bridge combination.

The combined effects of viscous damping and losses through the bridge result in a power spectral density that

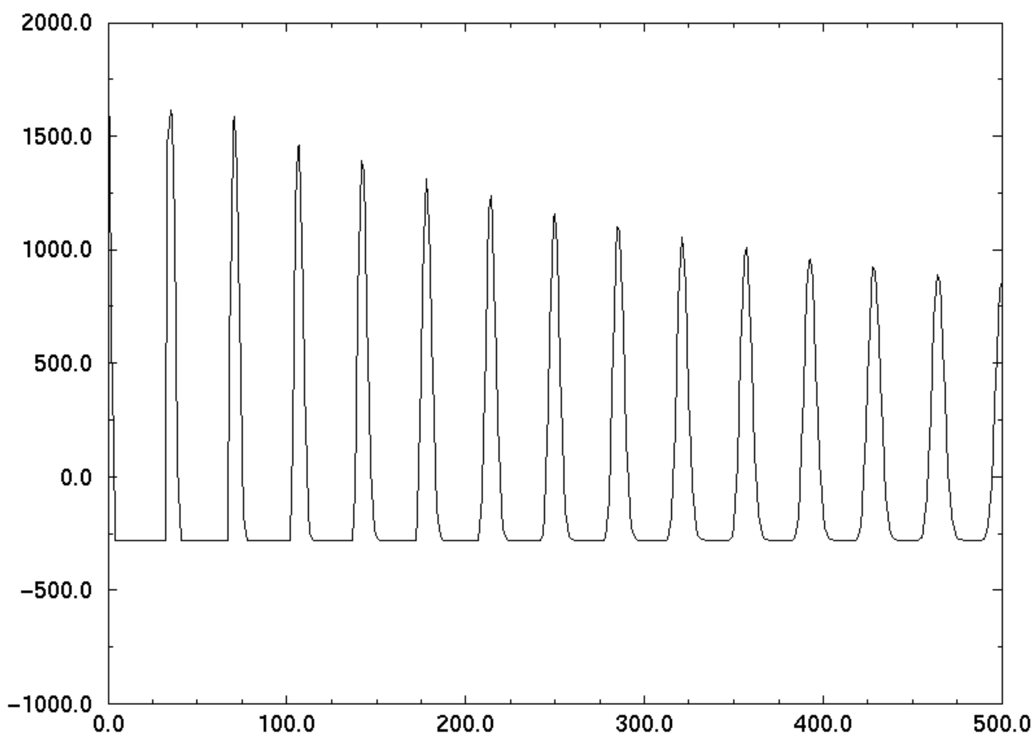

Figure 2: Waveshape of a plucked string near one endpoint including losses

is a function of time. The initial rectangular waveform described above is rich in harmonics. As time progresses, the amount of each harmonic decreases as a function of increasing frequency such that approximately one second after pluck only the fundamental remains.

The initial energy loss of the string is not sustainable. If the initial losses were due to viscous damping, the string would be completely damped before one second. Therefore, the initial “losses” imposed on the string are due to the excitation of resonate modes of the bridge and are not real losses. In effect, the bridge drives the string as well.

Presented in figure 1 is the plot of the power spectral density of the A string of an electric violin excited by a plucked stimulus. These plots show starting at the rear the initial spectral content of the forces applied at the feet of the bridge. As we move forward, we see that the fundamental progressively dominates. An important quality of a “good” violin is the duration of the “voice” or the degree to which the fundamental remains.

A string with fixed endpoints exhibits no time dependence with its power spectral density. This means that all the harmonics decrease with the same time dependence such that the mix remains constant. This also means that the rectangular waveform is maintained as time progresses. This would not be described as a pleasing sound quality.

3. COMPARISON OF REAL DATA WITH THE SIMULATION AND DIGITAL WAVEGUIDE

3.1. Simulation

The motion of a thin flexible string held under tension between fixed endpoints is described by a hyperbolic second order partial differential equation in one space dimension. The equation is given below for the undamped case.

$$C^2\frac{\delta^2 U}{\delta X^2} = \frac{\delta^2 U}{\delta^2 t}$$

If viscous damping is included the equation contains a term proportional to velocity. If either endpoint moves as a function of the forces generated by the motion of the string then the equation becomes more complex and may contain additional space dimensions.

In any investigation where we do not know the origin of the phenomena of interest nor the depth of the effects on the phenomena due to various motions of the string and its endpoints, it seems appropriate to begin with the lowest order of complexity and increase complexity only when necessary.

Since we do not have the luxury of extensive testing of real parameters, we have to employ a trial and error method of determining which physical parameters determine the sound of the plucked violin string and to what degree these or other parameters determine the quality of the sound. This is an impossible task but we have, none-the-less, undertaken it.

In order to maximize the possibility of a successful trial and error methodology, we opted to develop a “high fidelity” model of the string/endpoint physical system. The advantage to this method is that physical parameters such as mass, length, tension, stiffness as well as the spacial dimensions are employed directly in the model. Furthermore, these physical data are those most likely to be found in the literature. Conversely, finding the coefficients of some filter determined by earlier investigators to best model the bridge effects was considered to be much less likely.

The basis of this “high fidelity” model is the finite difference numerical method where-in the above wave equation is modeled by difference equations which relate the position of the nodes of the string at time (n+1)k and (m+1)h to the position of the nodes at nk and mh where nk is the current time step and mh is the current space step. Given initial conditions in time and space, the algorithm advances the position of the string to the desired time and space.

The explicit difference equation implied to advance the string motion for the conditions of viscous damping and rigid endpoints is given at the bottom of the page.

If other effects such as string stiffness or end point motion are to be included, the equations get considerably more complex and may involve additional space dimensions. It was determined at the onset that should such complexity be required that it would then place the “high fidelity” model outside the scope of the project.

$$U_m^{n+1} = \frac{1}{2+\frac{R}{E}k}\left[4\left(1-\frac{K}{E}p^2\right)U_m^n + 2\frac{K}{E}p^2(U^n_{m+1}+U^n_{m-1}) \div \left(\left(\frac{R}{E}k-2\right)U_m^{n-1}\right)\right]$$

Equation 1: DE for string motion and viscous losses

The initial level of complexity attempted was the model of a thin (vanishing diameter), flexible (zero stiffness), and heterogeneous string with rigid endpoints. This model, when plucked, generated a sound best described as a harmonic generator. The sound was not considered to be characteristic of the plucked violin string. The sound was not particularly pleasant either.

Thus additional complexity was required. Observation of the harmonic content versus time of the real data showed that the spectral components dissipated at a rate proportional to their frequency. Furthermore, the time constant was not single valued. The string energy decreased sharply after pluck but sustained for a period considerably longer than could be supported by the initial time constant. It appeared that the energy was being transferred to some other system from which the string could retrieve it later. The most likely adjoint system for energy storage and subsequent retrieval was the bridge. It was now apparent that the bridge must be modeled.

A search of the relevant literature turned up a mechanical model of the violin bridge. A “high fidelity” model of the bridge would be relatively complex. The bridge evidenced two degrees of freedom; it had a component of motion parallel to the string motion and a component normal to the string motion. These two motions were, unfortunately, coupled. A decision was made to decouple the motions such that a solution could be obtained without resorting to the complexity required of a system of simultaneous equations.

The new model including the decoupled bridge motions supported the assumption that the bridge was the sink/source for the string energy that we were looking for. However, there were serious technical problems with the simplified bridge motion. These problems created instabilities which caused the solution for the string not to converge unless additional damping, more than was desired, was applied to the system.Though the sound generated was clearly that of a plucked string; it was the sound of a severely damped string.

The next step would be to rework the algorithm into a system of simultaneous equations involving two space dimensions. This is, due to the considerable complexity involved, outside the scope of this effort.

String Model

Let us image a glass tube of arbitrary shape. Further, let us imagine a thin flexible string of mass density ε g per cm running through the tube. The string is spooled between two reels such that a tension T dynes is maintained. If this tube is not bent too sharply -which is to say the radius of curvature is large compared to the deviation of the tube from a straight line-the tension will be everywhere equal to T dynes (see figure 3).

The force against any inside portion of the tube wall is given by the relationship:

$$\Delta T = \phi T = \frac{T\Delta s}{R}$$

Given that the string is moving through the tube there will be a centrifugal force acting outward and therefore against the forces due to the string tension. This force is equal to the velocity squared times the mass of the portion of string under investigation divided by the radius of curvature. The net inward force is now:

$$\left(\frac{\Delta s}{R}\right)(T - \varepsilon\upsilon^2)$$

Since we can, in our imagination, run the string at any velocity we choose. Let us run the string at a velocity of

$$c = \sqrt{\frac{T}{\varepsilon}}$$

We notice that at velocity c, there are no forces acting against the tube walls! This means that if you break away the tube from around the string (you can do this in your imagination) the waveshape will remain unchanged!

The equivalent situation is when the string is stationary but the waveform moves along the string with velocity c independent of its shape.

The point of the above illustration is that the string shape remains unchanged as it travels along the string.

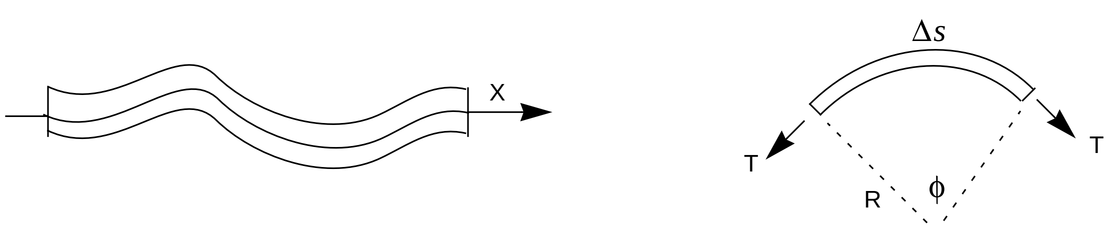

Figure 3: Forces on a moving string

Thus the general solution for the traveling wave must be of the form F(x-ct) where c is the velocity of propagation and the product ct represents a shifting along the x axis. Notice that shifting the wave in x does not change its shape just as shifting a time function in time (as Electrical Engineers are fond of doing) does not change its shape.

Its seems that we have only half of the answer; we must account for motion of the string in both directions. Therefore the general solution must be of the form

$$U = F(x - ct)$$

All the form of the general solution means is that wherever you see x you will see ct tagging along. We might refer to this property as (again borrowing from Electrical Engineering) “space invariance” in order to leverage off the term “time invariance” where-in the dependent variable t and its associated delay are always seen as pairs.

Now that we have this simple and elegant general solution, we wonder how we might find a specific solution given certain initial or boundary conditions. It seems logical to again tap the rich stores of Electrical Engineering theory and remember that any arbitrary wave shape can be synthesized as a superposition of harmonically related sinusoids. Thus, we can use the Fourier series as a synthesis tool by determining the spectral content of the initial string shape and realizing that shape will not change over our space variable x (assuming we have no losses and ideal boundary conditions), we can reflect the traveling waves at each endpoint and sum to obtain string displacement U as a function of time.

Again we are faced with the problem that though we have a simple and elegant solution, we can not use it. The above solution assumes no losses and ideal boundary conditions. We do not enjoy either of these conditions. Our problem includes viscous losses due to the string motion through the air, mechanical losses in the bridge and nut, and non-ideal boundary conditions as the bridge and (to a lesser degree) the nut move.

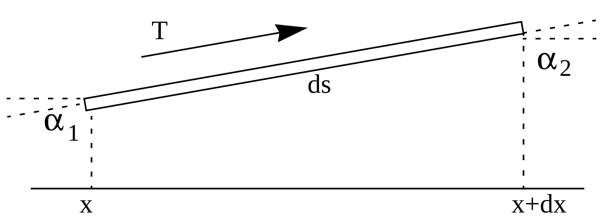

Figure 4: Forces of string under tension

Unfortunately, to find our next solution, we must leave the comforting confines of electrical engineering theory and enter the nether regions inhabited by mechanical engineers. We require our next solution in terms of forces, masses, lengths, moments and their relatives velocity and acceleration. Since we have to deal with the uncooperative air, bridge and nut we will have to work in their mechanical domain.

Wave Equation

Let us derive the wave equation. We shall again borrow from Morse (reference Vibration and Sound). We have a thin flexible string held between two end points under tension T dynes. We shall concern ourselves, however, with only a differential section ds of this string. This section of the string shall be displaced from the initial or unperturbed position as shown in figure 4.

The force perpendicular to the x axis is given by

$$T \sin\alpha_2 - T \sin\alpha_1$$

Since the αs are small (this is a restatement of the necessary condition that the radius of curvature is large compared to the deviation of the string about the x axis), we can employ the approximation:

$$\sin\alpha = \tan\alpha$$

Therefore, our force directing the string back to the x axis is given by

$$T(\tan\alpha_2 - \tan\alpha_1)$$

where

$$\tan\alpha = \frac{\partial U}{\partial x}$$

giving us

$$\tan\alpha_2 - \tan\alpha_1 = \frac{\partial U_{x+dx}}{\partial x} - \frac{\partial U_x}{\partial x}$$

We now have for the expression of force:

$$T\left[\frac{\partial U_{x+dx}}{\partial x} - \frac{\partial U_x}{\partial x}\right]$$

We can advance our derivation further if we remember that:

$$f(x+ds) = f(x) + dx\frac{\partial f(x)}{\partial x}$$

Now we have as our expression for force:

$$Tdx\left(\frac{\partial U_{x+dx}}{\partial x} - \frac{\partial U_x}{\partial x}\right) = Tdx\frac{\partial^2 U}{\partial x^2}$$

Since we have chosen a differential element and have applied a finite force to it, we must determine and apply an opposite force or wave bye to our differential element as it accelerates off to infinity. For our balancing act we will invoke the great wizard of Mechanical Land, Lagrange.

We know both the mass of our element and its velocity; therefore we can calculate the kinetic energy. We can safely ignore the potential energy due to localized overtension since we will assume that the tension is uniform along the string (actually the speed of propagation of the longitudinal wave is sufficiently fast that we can ignore localized overtension). Furthermore, if the tension is very high -and in our case it is, we can ignore gravitational forces. Therefore we have for our opposing force:

$$\frac{d}{dt}\left(\frac{\partial W_K}{\partial U_t}\right) + \frac{\partial}{\partial U}W_p$$

where

$$W_K = \frac{1}{2}\varepsilon ds(U_t^{\,2})$$

and

$$W_p \approx 0$$

Since we are discussing an element of vanishing length we can reasonably equate the differecial element ds with the differencial element dx. Thus we have:

$$\frac{d}{dt}\left[\frac{\partial}{\partial U_t}\left(\frac{1}{2}\varepsilon dx U_t^{\,2}\right)\right] = \varepsilon dx U_{tt}$$

or,

$$\varepsilon dx\frac{\partial^2 U}{\partial t^2}$$

Our opposing forces can now be equated:

$$Tdx\frac{\partial^2 U}{\partial x^2} = \varepsilon dx\frac{\partial^2 U}{\partial t^2}$$

or

$$\frac{T}{\varepsilon}\frac{\partial^2 U}{\partial x^2} = \frac{\partial^2 U}{\partial t^2}$$

where

$$\frac{T}{\varepsilon} = c^2$$

Finite Difference Methods

Now that we have the wave equation in terms of partial differential equations we can arrive at a unique solution reflecting our initial and boundary conditions. First we must recognize that our equation is of the type called hyperbolic. Given the following general equation:

$$a\frac{\partial^2 U}{\partial t^2} + 2b\frac{\partial^2 U}{\partial t^2 x^2} + c\frac{\partial^2 U}{\partial x^2} + d\frac{\partial U}{\partial t} + e\frac{\partial U}{\partial x} + fu = g$$

The equation is said to be hyperbolic if

$$b^2 - ac > 0$$

If we recall the definition of the derivative of f(x), we can begin construction of a numerical algorithm:

$$\lim_{x \to 0}\frac{f(x+k) - f(x)}{k} = f'(x)$$

For our purposes we will not allow h to approach zero and thus the reference to finite difference methods. Also, we will employ a “backward” difference along with the above forward difference. If we expand f(y,x + h) and f(y,x - h) as Taylor’s series, we can sum the two and solve for the second partial of f(y,x) with respect to x. Thus we have

$$f(y, x+k) = f(y, x) + k\frac{\partial}{\partial x}f(y, x) + \frac{k^2}{2}\frac{\partial^2}{\partial x^2}f(y, x)$$

and

$$f(y, x-k) = f(y, x) + -k\frac{\partial}{\partial x}f(y, x) + \frac{k^2}{2}\frac{\partial^2}{\partial x^2}f(y, x)$$

When we sum the above we have

$$f(y, x+k) + f(y, x-k) = 2f(y, x) + k^2 \frac{\partial^2}{\partial x^2} f(y, x)$$

and finally

$$\frac{\partial^2}{\partial x^2} f(y, x) = \frac{f(y, x+k) - 2f(y, x) + f(y, x-k)}{k^2}$$

If we perform the same manipulations on f(y,x) with regards the independent variable y, we will have our wave equation in terms of two difference equations: one in the space dimension y, and the other in the time dimension t.

Therefore, we have for y

$$\frac{\partial^2}{\partial y^2} f(y, t) = \frac{f(y+h, t) - 2f(y, t) + f(y-h, t)}{h^2}$$

The wave equation is now

$$c^2 \left[\frac{f(y+h, t) - 2f(y, t) + f(y-h, t)}{h^2} \right]$$

$$= \frac{f(y, t+k) - 2f(y, t) + f(y, t-k)}{k^2}$$

Substituting Um for f(y,t) , Um for f(y,t + k) , and Um+1 for f(y + h,t) we have

$$c^2 \left[\frac{U^n{}_{m+1} - 2U^n{}_m + U^n{}_{m-1}}{h^2} \right] = \frac{U_m{}^{n+1} + 2U_m{}^n + U_m{}^{n-1}}{k^2}$$

as our implicit difference equation.

Oh! But now you mention it. We forgot about losses; we have to account for viscous losses due to the string's motion in the air.

We now have to derive a difference equation for the first partial of displacement U with respect to time as the viscous losses are a function of the string's velocity.

We will use the same trick as above and we then have

$$\frac{\partial}{\partial t} f(y, t) = \frac{f(y, t+k) - f(y, t-k)}{2k}$$

We must now add the new term to the original equation. We now have

$$c^2 \left[\frac{U^n{}_{m+1} - 2U^n{}_m + U^n{}_{m-1}}{h^2} \right] = \frac{U_m{}^{n+1} + 2U_m{}^n + U_m{}^{n-1}}{k^2}$$

$$R\left(\frac{U_m{}^{n+1} + U_m{}^{n-1}}{2k} \right)$$

Our difference equation becomes

$$U_m{}^{n+1} = \frac{1}{2 + \frac{R}{\varepsilon}k} [Z]$$

$$Z = \left[4\left(1 - \frac{k}{\varepsilon}\rho^2\right) U_m{}^n + 2\frac{k}{\varepsilon}\rho^2 \left(U_{m-1}{}^n + U_{m+1}{}^n \right) + Y \right]$$

$$Y = \left(\frac{R}{\varepsilon}k - 2 \right) U_m{}^{n-1}$$

Where, $c = \sqrt{\frac{k}{\varepsilon}}$ and R is the damping coefficient.

Ah! But we still have a problem. We have our initial conditions and our boundary conditions. We also have a method of advancing to U at n+1 given U at n so you ask what can be the problem. If you will notice, our algorithm requires information not just at n but also at n-1.

If we start at n=0, n-1 is undefined. Therefore we must start at n=1. But we have no knowledge of the solution at n=1. We therefore require an auxiliary equation to solve for U at n=1 based on knowledge of U at n=0.

The solution for the auxiliary equation must not contain any terms at n-1. Therefore if we expand f(y,t + k) in a Taylor series we have

$$(y,t+k) = f(y,t) + k\frac{\partial}{\partial t} f(x,t) + \frac{k^2}{2} \frac{\partial^2}{\partial t^2} f(x,$$

If we remember that

$$\frac{\partial^2}{t^2} f(y,t) = c^2 \frac{\partial^2}{\partial y^2} f(y,t) - R\frac{\partial}{\partial t} f(y,t$$

We can make the substitution

$$(y,t+k) = f(y,t) + k\frac{\partial}{\partial t} f(x,t) + \frac{k^2}{2}\left[c^2 \frac{\partial^2}{\partial y^2} f(y,t) - R\frac{\partial}{\partial t} f(y,t) \right.$$

We need now only one more piece of information. At t=0- we should expect the string velocity to be zero. If we then discard the velocity term in the above equation, we will have

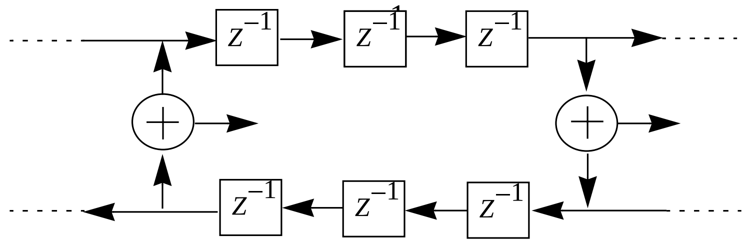

Figure 5: Simple string with rigid endpoints

$$f(y,t+k) = f(y,t) + \frac{k^2c^2}{2}\frac{\partial^2}{\partial y^2}f(y,t)$$

It follows then that

$$U_m^{n+1} = U_m^n + \frac{k^2c^2}{2}\left[\frac{U_{m+1}^n + 2U_m^n - U_{m-1}^n}{h^2}\right]$$

Thus we have our auxiliary equation which will advance our solution to n=1 from which our main equation will advance the solution to it's ultimate destination.

Non-Rigid Endpoints

For the reader who is rabidly hanging on each word, I apologize. The above bold heading promising information about moving endpoints was a tease. For those readers trudging ahead on shear determination, your ordeal is over. Well! One more paragraph.

Though we made a valiant effort, we never succeeded in generating an accurate model of the string attached to the objective movable endpoint -the bridge. Some success was had with simplified moving endpoints but they were of no interest as regards the generation of "violin like" sounds. The effort has, by no means, been abandoned.

3.2. The Digital Waveguide

The numerical method for computing the motion and forces of a plucked string is thorough, but lacks the desired response speed. A simulation containing only one pluck could take up to an hour to process. The same mathematical analysis can be used to derive a method for performing the same tasks with much less strain on a CPU. A string can be modeled as two parallel delay lines. One delay line can be used to carry the waves travelling in one direction, while the waves which bounce off of the nut and travel in the opposite direction can be carried by the other delay line. This is illustrated in figure 5. The output can be found at any point along the string by summing the values of the upper and lower rails at that point.

In the simplest case, a string with two rigid supports can be modeled with two delay lines, and inverters at each end. The inverters invert the wave as it bounces off of the nut, as would a real wave. The initial value of the displacement of each node can be calculated knowing the point of maximum displacement and the amount of that displacement.

A more accurate representation of a violin string can be represented in figure 6. In this case, the nut is considered to be a rigid support, and the bridge is not. The losses at the bridge can be modeled with a lowpass filter. because of the differences in the physical characteristics in each string on the violin, each string will have its own transfer curve. The strings which produce higher pitches are under more tension, and have less mass. This causes

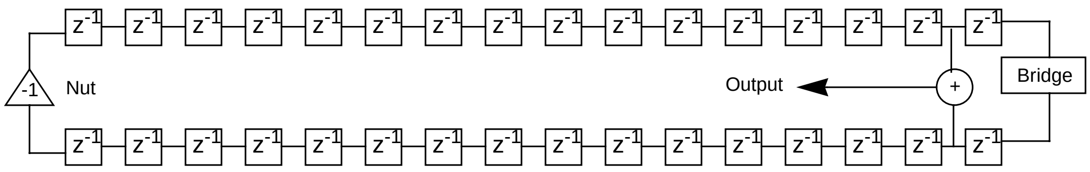

Figure 6: Digital waveguide model of string with non-rigid endpoints

them to be less affected by the bridge. In other words, for smaller strings with less mass, the bridge appears to be more rigid.

This was easily shown with the digital waveguide model. The initial filter to be implemented was a singe pole FIR:

$$u(n) = \frac{1}{2} + \frac{1}{2}z^{-1}$$

This filter gave the more massive, lower frequency strings an abnormally long decay time. The higher strings appeared normal, however. This was remedied by implementing a higher order filter to the lower strings. The filter used for the G and D strings is a two pole FIR:

$$u(n) = \frac{1}{2} + \frac{1}{4}z^{-1} + \frac{1}{4}z^{-2}$$

While these filters are not true representations of the process by which the high frequencies are attenuated at the bridge, it does provide a more accurate picture than a single transfer function for all four strings. If a solution to the wave equation model had been derived, that information could have been directly transferred to the digital waveguide model, just as the string model was derived.

4. EVALUATION

The data produced by the digital waveguide model was able to closely approximate these characteristics. An exact reproduction of the characteristics was not able to be achieved due to the complexity of the filter that would be necessary to get the exact characteristics. We were able to produce data from this model which had high attenuation rates at high frequencies similarly to the real data. Although the high-frequency components of this data were not that rich in harmonics of their counterparts, the actual sound output by this data closely matched with that of a real violin.

The observation of the plots of the data produced by the numerical solution model showed that all the frequencies had the same attenuation rate. This was due to the fact that we were unable to incorporate the exact model of a bridge into the model, as a solution of the behavior of the bridge was too complicated

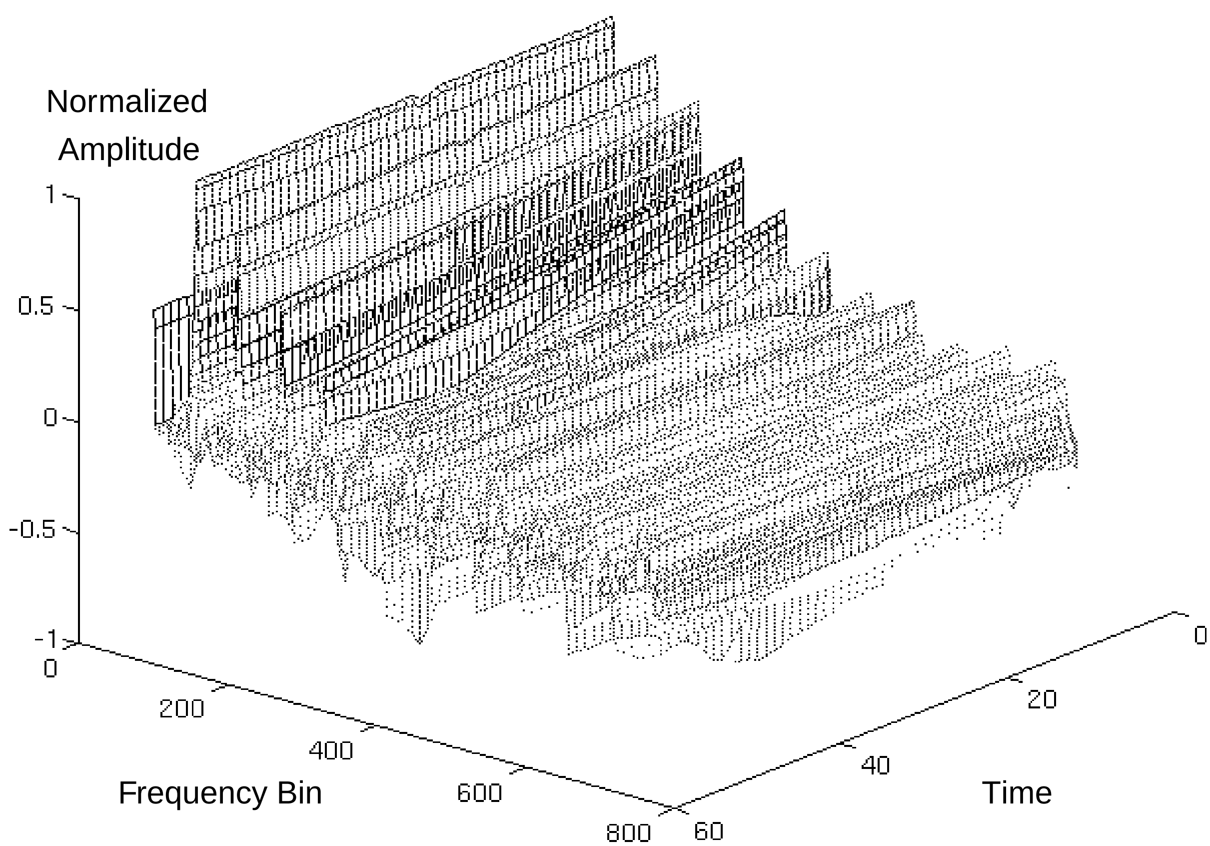

Figure 7: Harmonic spectrum of digital waveguide model incorporating losses at bridge

mathematically to determine. The bridge gives rise to the various attenuation rates of the frequency components and it's absence should meant that all the should be having the same attenuation (or no attenuation), which is exactly what was observed from the plot.

5. TCL INTERFACE

Any high fidelity physical model is a complex system. Even liberal comments do not make the code user friendly. Thus to promote general access, we have implemented a point-and-click interface.

When designing the GUI, we had in mind the wide range of users that might want to use this system. It was important that anyone could use even those that are not adept in using Unix tools. Users do not what to have to try to interpret someone else's code to be able to achieve the sound produced by the digital-waveguide. With this in mind, we began to lay out a design of how we wanted the interface to interpret our data and present to the user in a useful manner.

The first step was to present to the user four input options that they could perform. These included looking at the figure and a single string of the instrument where the user is allowed to pluck the string and watch what the strings does after being plucked. The next option is to be able to choose a specific note, string, and either sharp, flat, or natural, which is then either played or stored in an output file. The note that is heard is produced from the digital waveguide in real time. The third and fourth options are to choose a file to play or to connect to a MIDI interface.

6. SUMMARY

This project successfully completed its task of developing a real-time demonstration involving physical modeling. Although the end product, at this point, is not as complete a model as would be desired, the model does exhibit the sonic characteristics of a plucked violin string.

This project showed the importance of approaching the problem from both the wave equation and digital waveguide methods. The wave equation method, though computationally intensive, produces output which is mathematically accurate. Because Matlab was used, once the mathematical expression was derived, the solution could then be directly solved. This provides for an environment where

The digital waveguide is much better suited for implementation in a real time system because it consists of many functional blocks which are very simple. This provides for an environment where development of the model can be done purely mathematically, and once the mathematical model is complete, it can be applied to the digital waveguide model for implementation into a live application.

The chief limitation of our project is the failure to develop a model of the bridge in the wave equation model. Though a mathematical model was not developed, experimentation with the digital waveguide model showed that sufficient results could be obtained from a system of low-order low pass filters.

Further work could be done to develop the bowed string model and the model of the resonant cavity. Completion of this would produce a complete model of the violin.

REFERENCES

1. Beranek, Leo L., *Acoustics*, American Institute of Physics.
2. Bladier, Bejamin, “On the bridge of the Violincello,” Compt. Rend. Vol 250, pp 2161 - 2163. March 1960.
3. Campbell, Murray, and Greated, Clive, *Musician’s Guide to Acoustics*, Schirmer Books, Ney York, 1987.
4. Childers, Donald and Durling, Allen, *Digital Filtering and Signal Processing*, West Publishing Co.
5. Cremer, Lothar, “Influence of Bow Pressure on the Movement of a Bowed String,” Catgut Acoustical Society Newsletter, No 18, pp 13-19. 1972.
6. Cremer, Lothar, *Physics of the Violin*, The MIT Press, Cambridge, Massachussetts, 1981.
7. Gear, C. William, *Numerical Initial Value Problems in Ordinary Differential Equations*, Pretice Hall.
8. Hutchins, C. M., “The Physics of Violins”, *Sientific American*, pp 78-93. November 1962
9. Itowa, Hideo, and Chhiro Kumagai, “On the Study of Violins and a Scientific Basis for Violin Construction,” *Rept. Inst. Ind. Sci. Univ. Tokyo*,vol 3, No. 1, pp 5-19. 1957.
10. Kincaid, Cheny, *Numerical Analysis*, Brooks/Cole Publishing Co.
11. Mcintyre, M. E., Schumacher, R. T., and Woodhouse, J., “On the Oscillations of Musical Instruments,” *Journal of the Acoustical Society of America*, vol 74, No. 5, pp 1325-45. November 1983
12. Morse, P.M., *Vibration and Sound*, American Institute of Physics for the Acoustical Society of America, Woodbury, New York, USA, 1976.
13. Ohlson, Harry F. *Elements of Acoustical Engineering*, D. Van Nostrand Co., Inc., New York, New York, 1947
14. Reid, Christopher E. and Thomas B. Passin, *Signal Processing in C*, John Wiley & Sons, Inc., New York, 1992.
15. Rorabaugh, C. Britton, *Digital Filter Designer’s Handbook: Featuring C Routines*, Mc Graw - Hill, Inc., Blue Ridge Summit, PA, 1993.
16. Smith III, Julius O., “Physical Modeling Using digital Waveguides,” *Computer Music Journal*, vol. 16, No. 4, pp 74-91. Winter 1992.
17. Swisher, George M., *Introduction to Linear Systems Analysis*, Matrix Publishers, Inc.
18. Watkinson, John, *The Art of Digital Audio*, Focal Press, London, 1988.