Locations of the Zeros of a Linear Phase FIlter

An FIR filter can be described by a difference equation:

$$y(n) = \sum_{k=0}^{M-1} h(k)x(n-k) \quad \text{or,} \quad H(z) = \sum_{k=0}^{M-1} h(k)z^{-k}$$

If the filter is a linear phase filter, its impulse response MUST satisfy the constraint:

$$h(n) = \pm h(M-1-n) \qquad n = 0, 1, \ldots, M-1$$

"+" corresponds to the symmetry case, "-" corresponds to antisymmetry. We can compactly represent the frequency response as:

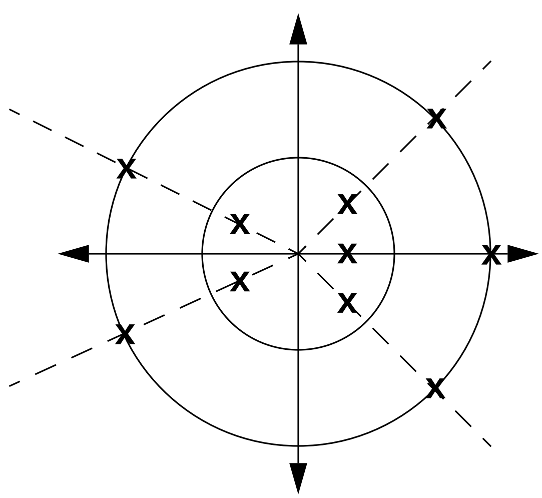
$$m \text{ odd: } H(z) = z^{-(M-1)/2}\left\{h\left(\frac{M-1}{2}\right) + \sum_{n=0}^{(M-3)/2} h(n)[z^{(M-1-2n)/2} \pm z^{-(M-1-2n)/2}]\right\}$$

$$m \text{ even: } H(z) = z^{-(M-1)/2}\left\{\sum_{n=0}^{(M/2)-1} h(n)[z^{(M-1-2n)/2} \pm z^{-(M-1-2n)/2}]\right\}$$

If we substitute z^{-1} for z, and multiply both sides by $z^{-(M-1)}$, we obtain:

$$z^{-(M-1)}H(z^{-1}) = \pm H(z)$$

This implies that the roots of $H(z)$ occur in reciprocal pairs, and conjugate pairs if $h(n)$ has real coefficients:



Design of Linear-Phase FIR Filters Using Windows

Suppose we want to design a linear phase lowpass FIR filter:

$$H_d(\omega) = \begin{cases} 1e^{-j\omega(M-1)/2} & 0 \le \omega \le \omega_c \\ 0 & otherwise \end{cases}$$

We can compute $h_d(n)$ using the inverse transform:

$$h_d(n) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega\left(n-\frac{M-1}{2}\right)} d\omega$$

$$= \frac{\sin\omega_c\left(n-\frac{M-1}{2}\right)}{\pi\left(n-\frac{M-1}{2}\right)}$$

Clearly, $h_d(n)$ is noncausal and infinite in duration. We can truncate using a window:

$$h(n) = h_d(n)w(n)$$

What are the drawbacks of this approach?

Can we generalize this?

Hamming window: $0.54 - 0.46\cos\left(\frac{2\pi n}{M-1}\right)$

Hanning window: $\frac{1}{2}\left(1-\cos\left(\frac{2\pi n}{M-1}\right)\right)$

Kaiser window: $$\frac{I_0\left[\alpha\sqrt{\left(\frac{M-1}{2}\right)^2-\left(n-\frac{M-1}{2}\right)^2}\right]}{I_0\left[\alpha\left(\frac{M-1}{2}\right)\right]}$$

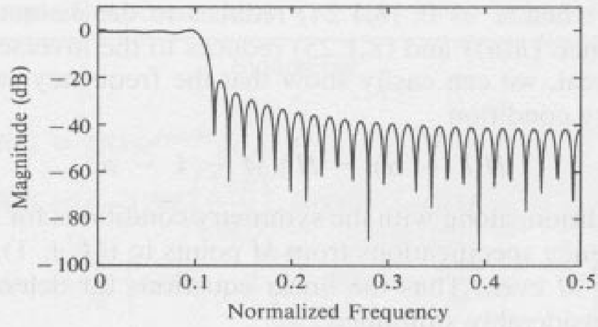


FIGURE 8.8 Lowpass FIR filter designed with rectangular window ($M = 61$).

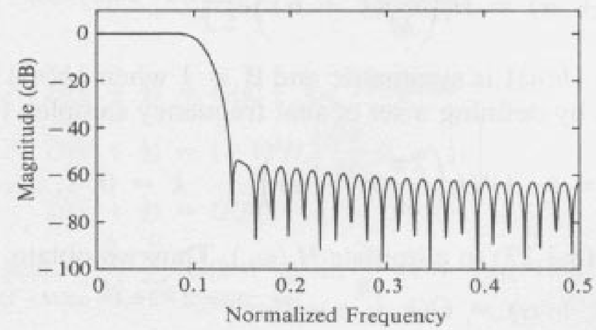


FIGURE 8.9 Lowpass FIR filter designed with Hamming window ($M = 61$).

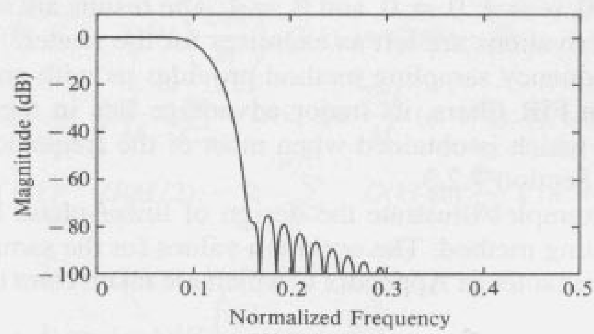


FIGURE 8.10 Lowpass FIR filter designed with Blackman window ($M = 61$).

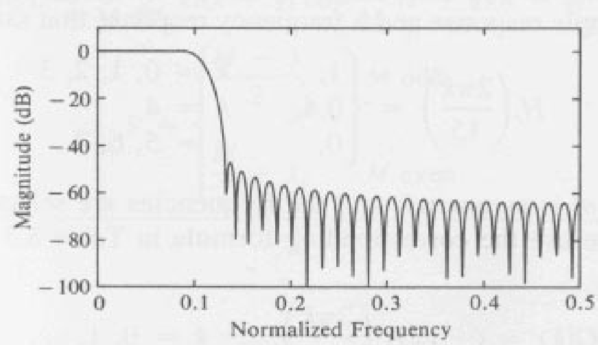


FIGURE 8.11 Lowpass FIR filter designed with $\alpha = 4$ Kaiser window ($M = 61$).

Design of Linear-Phase FIR Filters By Frequency Sampling Methods

Let $H_d(\omega)$ be specified by an equispaced set of samples:

$$H_d(k+\alpha) \equiv H_d(\omega)\Big|_{\omega = \frac{2\pi}{M}(k+\alpha)} \qquad k = 0, 1, \ldots, \frac{M-1}{2} \qquad (M\ odd)$$

$$k = 0, 1, \ldots, \frac{M}{2} - 1 \qquad (M\ even)$$

Then, we can compute the filter impulse response from the inverse transform:

$$h(n) = \frac{1}{M}\sum_{k=0}^{M-1} H_d(k+\alpha)e^{j2\pi(k+\alpha)n/M}$$

Since $\{h(n)\}$ are real, we can show:

$$H(k+\alpha) = H^*(M-k-\alpha)$$

By defining a set of real frequency samples, $\{G(k+\alpha)\}$, we can simplify this design approach as follows:

$$G(k+\alpha) = (-1)^k\left|H_d\left(\frac{2\pi}{M}(k+\alpha)\right)\right| \qquad k = 0, 1, \ldots, M-1,$$

we can show:

$$H(k+\alpha) = G(k+\alpha)e^{j\pi k}e^{j[\beta\pi/2 - 2\pi(k+\alpha)(M-1)/2M]}$$

This covers four cases (see Table 8.3):

- symmetric $\beta = 0$)/antisymmetric ($\beta = 1$)
- $\alpha = 0/\alpha = 1$

Why is this still not a useful design methodology?

Design of Optimum Equiripple Linear-Phase FIR Filters

Consider the problem:

$$1-\delta_1 \le H_r(\omega) \le 1+\delta_1 \qquad |\omega| \le \omega_p$$
$$-\delta_2 \le H_r(\omega) \le \delta_2 \qquad |\omega| \ge \omega_s$$

Suppose we constrain our choices for possible filters to a linear phase filter where $h(n) = h(M-1-n)$, and M is odd. Then,

$$H_r(\omega) = h\left(\frac{M-1}{2}\right) + 2\sum_{n=0}^{(M-3)/2} h(n)\cos\left(\omega\left(\frac{M-1}{2}-n\right)\right)$$

Let $k = \frac{M-1}{2} - n$, and:

$$a(k) = \begin{cases} h\left(\frac{M-1}{2}\right) & k = 0 \\ 2h\left(\frac{M-1}{2}-k\right) & k = 1, 2, \ldots, \frac{M-1}{2} \end{cases}$$

Using these definitions,

$$H_r(\omega) = \sum_{k=0}^{(M-1)/2} a(k)\cos\omega k$$

Our strategy is to solve for $\{a(k)\}$ from $H_r(\omega)$, and then $h(n)$ from $\{a(k)\}$.

Let us add an optimization component to the problem (let the user decide what aspects of the design are important):

$$H_r(\omega) = Q(\omega)P(\omega)$$

where $Q(\omega) = 1$ and $P(\omega) = \sum_{k=0}^{L} a(k)\cos\omega k$

Define a real-valued weighting function:

$$H_{dr}(\omega) = \begin{cases} 1 & |\omega| \le \omega_p \\ 0 & |\omega| \ge \omega_s \end{cases} \quad \text{and} \quad W(\omega) = \begin{cases} \delta_2/\delta_1 & |\omega| \le \omega_p \\ 0 & |\omega| \ge \omega_s \end{cases}$$

The weighted error can be expressed as:

$$E(\omega) = W(\omega)[H_{dr}(\omega) - H_r(\omega)]$$
$$= W(\omega)Q(\omega)[H_{dr}(\omega)/Q(\omega) - P(\omega)]$$

or,

$$E(\omega) = \hat{W}(\omega)[\hat{H}_{dr}(\omega) - P(\omega)]$$

where,

$$\hat{W}(\omega) = W(\omega)Q(\omega)$$
$$\hat{H}_{dr}(\omega) = H_{dr}(\omega)/Q(\omega)$$

We would like a procedure to minimize $E(\omega)$:

Alternation Theorem: *Let S be a compact subset of the interval $[0, \pi]$. A necessary and sufficient condition for:*

$$P(\omega) = \sum_{k=0}^{L} a(k)\cos\omega k$$

to be the unique, best-weighted Chebyshev approximation to $\hat{H}_{dr}(\omega)$ in S is that the error function $E(\omega)$ exhibit at least $L+2$ extremal frequencies in S. That is, there must exist at least $L+2$ frequencies $\{\omega_i\}$ in S such that:

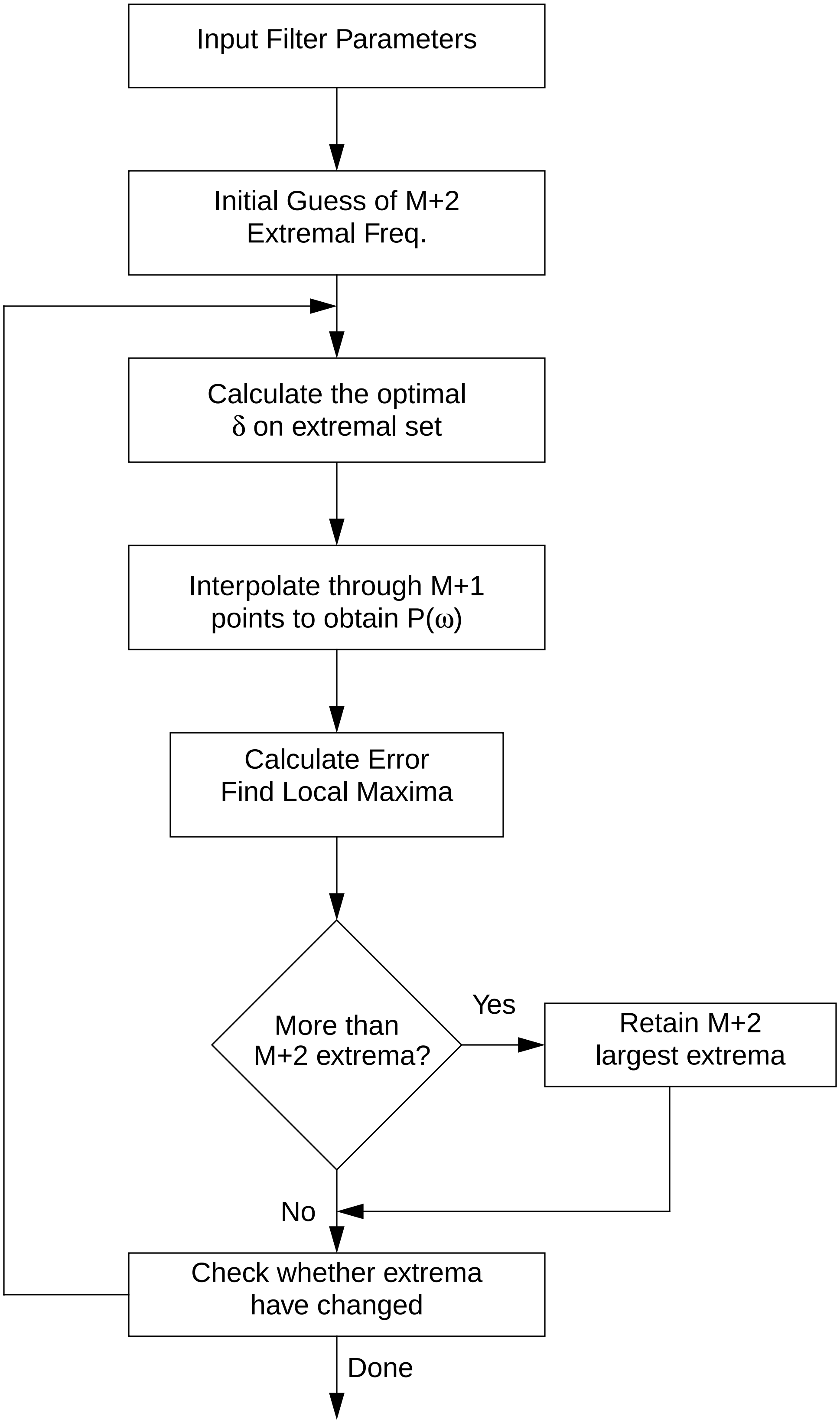
$$\omega_1 \le \omega_2 \le \ldots \le \omega_{L+2}$$

$$E(\omega_i) = -E(\omega_{i+1})$$

$$|E(\omega_i)| = \max_{\omega \in S} |E(\omega)| \qquad i = 1, 2, \ldots, L+2$$

$E(\omega)$ alternates in sign between a maximum and minimum, hence the theorem is called the alternation theorem. Several procedures exist to find $P(\omega)$. The most famous is the Remez exchange algorithm:

An Overview of the Remez Exchange Algorithm



Parameters Of The Parks-McLellan Program

NFILT: The filter length, denoted above as M.

JYTPE: The type of filter:
- JTYPE=1 results in a multiple passband/stopband filter.
- JTYPE=2 results in a differentiator.
- JTYPE=1 results in a Hilbert transformer.

NBANDS: The number of frequency bands (typically ranges from 2 for a lowpass to a software-dependent maximum for a multiple-band filter).

LGRID: The grid density for interpolating the error function (usually 16 by default).

EDGE: Lower and upper cutoff frequencies of the bands.

FX: Desired frequency response of each band (band gain).

WTX: Weight function in each band.

This algorithm can be found embedded in many tools, including Matlab.

What is wrong with this approach?

Frequency Transformations For Analog Filters

Type of Transformation	Transformation	Cutoff Frequencies of New Filter
Lowpass	$s \rightarrow \frac{\Omega_p}{\Omega'_p} s$	Ω'_p
Highpass	$s \rightarrow \frac{\Omega_p \Omega'_p}{s}$	Ω'_p
Bandpass	$s \rightarrow \Omega_p \left(\frac{s^2 + \Omega_l \Omega_u}{s(\Omega_u - \Omega_l)} \right)$	Ω_l, Ω_u
Bandstop	$s \rightarrow \Omega_p \left(\frac{s(\Omega_u - \Omega_l)}{s^2 + \Omega_l \Omega_u} \right)$	Ω_l, Ω_u

Strategy:

(1) Prewarp the cutoff frequencies
(2) Design an analog lowpass filter
(3) Use a frequency transformation to prewarped frequencies
(4) Use the bilinear transform to get a digital filter

Frequency Transformations For Digital Filters

Type of Transformation	Transformation	Cutoff Frequencies of New Filter
Lowpass	$z^{-1} \to \frac{z^{-1} - a}{1 - az^{-1}}$	ω'_p $a = \frac{\sin[(\omega_p - \omega'_p)/2]}{\sin[(\omega_p + \omega'_p)/2]}$
Highpass	$z^{-1} \to -\frac{z^{-1} + a}{1 + az^{-1}}$	ω'_p $a = -\frac{\cos[(\omega_p + \omega'_p)/2]}{\cos[(\omega_p - \omega'_p)/2]}$
Bandpass	$z^{-1} \to -\frac{z^{-2} - a_1 z^{-1} + a_2}{a_2 z^{-2} - a_1 z^{-1} + 1}$	ω_l, ω_u $a_1 = -2\alpha K/(K+1)$ $a_2 = (K-1)/(K+1)$ $\alpha = -\frac{\cos[(\omega_u + \omega_l)/2]}{\cos[(\omega_u - \omega_l)/2]}$ $K = \cot\left(\frac{\omega_u - \omega_l}{2}\right)\tan\left(\frac{\omega_p}{2}\right)$
Bandstop	$z^{-1} \to \frac{z^{-2} - a_1 z^{-1} + a_2}{a_2 z^{-2} - a_1 z^{-1} + 1}$	ω_l, ω_u $a_1 = -2\alpha/(K+1)$ $a_2 = -(K-1)/(K+1)$ $\alpha = \frac{\cos[(\omega_u + \omega_l)/2]}{\cos[(\omega_u - \omega_l)/2]}$ $K = \tan\left(\frac{\omega_u - \omega_l}{2}\right)\tan\left(\frac{\omega_p}{2}\right)$

Strategy:

(1) Design a digital lowpass filter (using standard techniques)

(2) Use a digital frequency transformation