Relationship of the DFT to Other Transforms

- A periodic sequence $\{x_p(n)\}$ with fundamental period N can be represented in a Fourier series of the form:

$$x_p(n) = \sum_{n=-\infty}^{\infty} c_k e^{j2\pi kn/N}, \qquad -\infty < n < \infty$$

where the Fourier series coefficients are given by the expression

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x_p(n) e^{-j2\pi kn/N}, \qquad k = 0, 1, 2, \ldots, N-1$$

The DFT is therefore related to the Fourier Series by the simple expression:

$$X(k) = Nc_k$$

- For an aperiodic sequence, $\{x(n)\}$ we have shown that:

$$X(k) = X(\omega)\big|_{\omega = 2\pi k/N} = \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi kn/N}, \qquad k = 0, 1, 2, \ldots, N-1$$

are the DFT coefficients of the periodic sequence of period N, given by

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n - lN)$$

and $x_p(n)$ is determined by aliasing $x(n)$ over the interval $0 \le n \le N-1$. The finite duration sequence

$$\hat{x}(n) = \begin{cases} x_p(n) & 0 \le n \le N-1 \\ 0 & otherwise \end{cases}$$

is obviously different from the original sequence unless $x(n)$ is finite duration and length $L \le N$, in which case

$$\hat{x}(n) = x_p(n), \qquad 0 \le n \le N-1$$

Only in this case will the IDFT return $x(n)$.

- Recall the z-Transform:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

If we sample this transform at equally spaced points on the unit circle,

$$X(k) \equiv X(z)\Big|_{z = e^{j2\pi k/N}} = \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi kn/N}, \qquad k = 0, 1, 2, \ldots, N-1$$

If $x(n)$ is of finite duration of length N, its z-transform is uniquely determined by an N-point DFT:

$$X(z) = \sum_{n=0}^{N-1} x(n) z^{-n}$$

$$X(z) = \sum_{n=0}^{N-1} \left[\frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j2\pi kn/N} \right] z^{-n}$$

which can be simplified to:

$$X(z) = \frac{1 - z^{-N}}{N} \sum_{k=0}^{N-1} \frac{X(k)}{1 - e^{j2\pi k/N} z^{-1}}$$

This expression is similar to the one we derived for the Fourier transform:

$$X(\omega) = \frac{1 - e^{-j\omega N}}{N} \sum_{k=0}^{N-1} \frac{X(k)}{1 - e^{-j(\omega - 2\pi k)/N}}$$

or,

$$X(\omega) = \sum_{k=0}^{N-1} X\left(\frac{2\pi}{N} k\right) P\left(\omega - \frac{2\pi}{N} k\right), \qquad N \geq L$$

- Suppose that $x_a(t)$ is a continuous-time periodic signal with a fundamental period $T_p = 1/F_o$. The signal can be expressed in a Fourier Series

$$x_a(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k F_0 t}$$

If we sample $x_a(t)$ at a uniform rate $F_s = N/T_p = 1/T$, we can show that the Fourier series coefficients are related:

$$X(k) = N \sum_{l=-\infty}^{\infty} c_{k-lN} \equiv N\tilde{c}_k$$

These are an aliased version of the Fourier series coefficients $\{c_k\}$.

- Recall the relationship of the spectrum of a discrete signal to the Fourier transform of a continuous-time signal:

$$X\left(\frac{F}{F_s}\right) = F_s \sum_{m=-\infty}^{\infty} X_a(F - mF_s), \quad \text{or,} \quad X(\omega) = F_s \sum_{m=-\infty}^{\infty} X_a([\omega - 2\pi m]F_s)$$

We can easily show that:

$$X(k) = F_s \sum_{m=-\infty}^{\infty} X_a\left(\frac{kF_s}{N} - mF_s\right)$$

and,

$$\sum_{l=-\infty}^{\infty} x_a(nT - lNT) \leftrightarrow F_s \sum_{m=-\infty}^{\infty} X_a\left(\frac{kF_s}{N} - mF_s\right)$$

This last relation illustrates time-domain and frequency-domain aliasing.

Time-Domain Windowing

Let $\{x(n)\}$ denote a sequence to be analyzed. Let's limit the duration of $\{x(n)\}$ to L samples:

$$\hat{x}(n) = x(n)w(n)$$

where $w(n)$ is a rectangular window and is defined as

$$w(n) = \begin{cases} 1, & 0 \le n \le L-1 \\ 0, & otherwise \end{cases}.$$

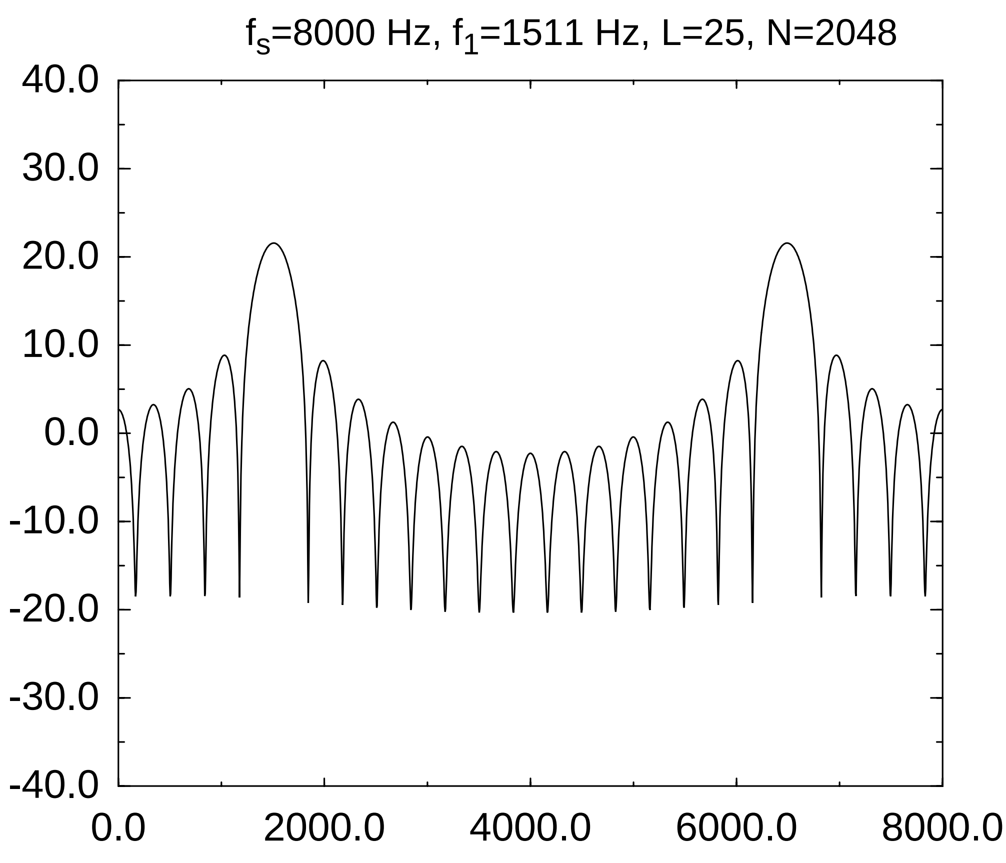
The Fourier transform of $w(n)$ is given by:

$$W(\omega) = \frac{\sin(\omega(L/2))}{\sin(\omega/2)} e^{-j\omega((L-1)/2)}$$

The transform of $\hat{x}(n)$ is given by:

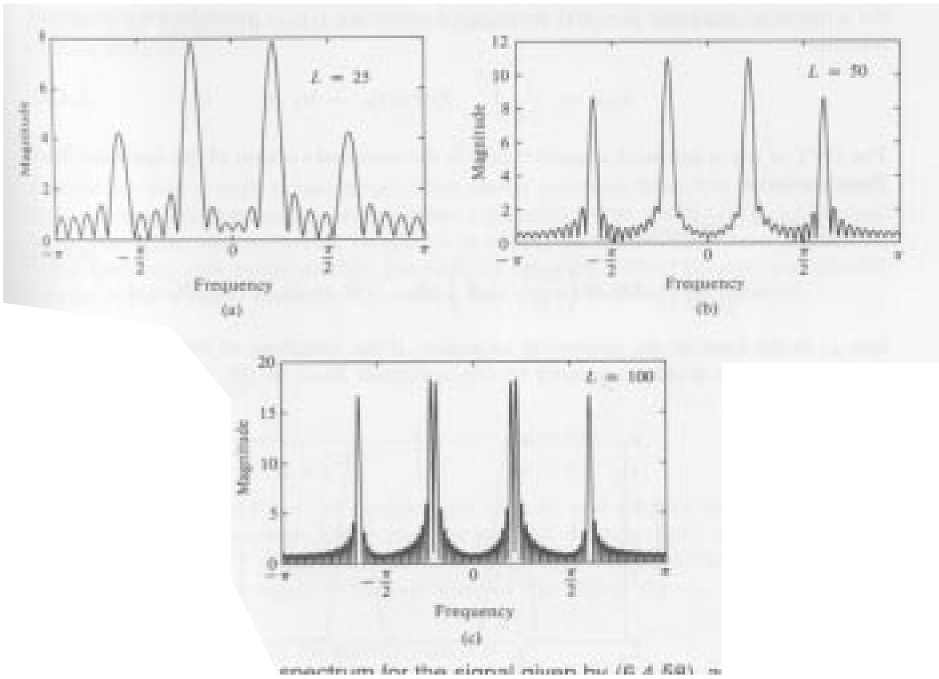
$$\hat{X}(\omega) = \frac{1}{2}[W(\omega-\omega_o) + W(\omega+\omega_o)].$$

This introduces frequency domain aliasing (the so-called picket fence effect):

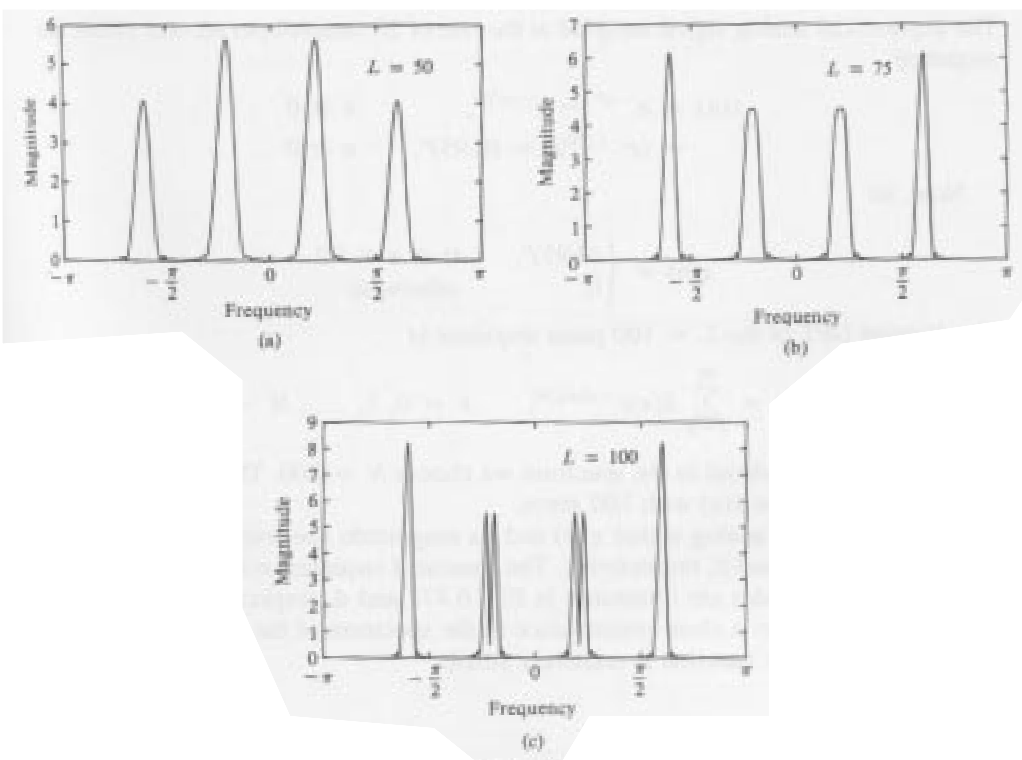


Improvements Via Better Windows

Rectangular Window:



Hanning Window:



Popular Windows

1. Rectangular:

$$w(k) = \begin{cases} 1, & |k| \le N \\ 0, & otherwise \end{cases}$$

2. Generalized Hanning:

$$w_H(k) = w(k)\left[\alpha + (1-\alpha)\cos\left(\frac{2\pi}{N}k\right)\right] \qquad 0 < \alpha < 1$$

$$\alpha = 0.54, \qquad Hamming\ window$$
$$\alpha = 0.50, \qquad Hanning\ window$$

3. Bartlett

$$w_B(k) = w(k)\left[1 - \frac{|k|}{N+1}\right]$$

4. Kaiser

$$w_K(k) = w(k) I_0\left(\alpha\sqrt{1 - \frac{K^2}{N}}\right) / I_0(\alpha)$$

5. Chebyshev:

$$w_N(k) = 2(x_0^2 - 1) w_{N-1}(k) + x_0^2[w_{N-1}(k-1) + w_{N-1}(k+1)] - w_{N-2}(k)$$

6. Gaussian

$$w_G(k) = \begin{cases} \exp\left[-\frac{1}{2}k^2 \tan^2\left(\frac{\theta_0}{2}\right)\right] & |k| < N \\ w_G(N-1) / \left[2N\sin^2\left(\frac{\theta_0}{2}\right)\right] & |k| < N \\ 0 & |k| > N \end{cases}$$

There are many others. The most important characteristics are the width of the main lobe and the attenuation in the stop-band (height of highest sidelobe). The Hamming window is used quite extensively.