

## Frequency-Domain Characteristics of Linear Time-Invariant Systems

Response to Complex Exponential and Sinusoidal Signals:

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

$$x(n) = Ae^{j\omega n}$$

$$\begin{aligned} y(n) &= \sum_{k=-\infty}^{\infty} h(k)[Ae^{j\omega(n-k)}] \\ &= A \left[ \sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k} \right] e^{j\omega n} \end{aligned}$$

Note that the term in brackets is the Fourier Transform of the impulse response:

$$H(\omega) = \sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k}$$

$H(\omega)$  exists if the system is BIBO stable:

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

Hence, the response of the system to a complex exponential is given by:

$$y(n) = AH(\omega)e^{j\omega n}$$

Note that if I vary  $\omega$ , I can compute the frequency response.  
Suppose I apply a “chirp” function?

**Example:**

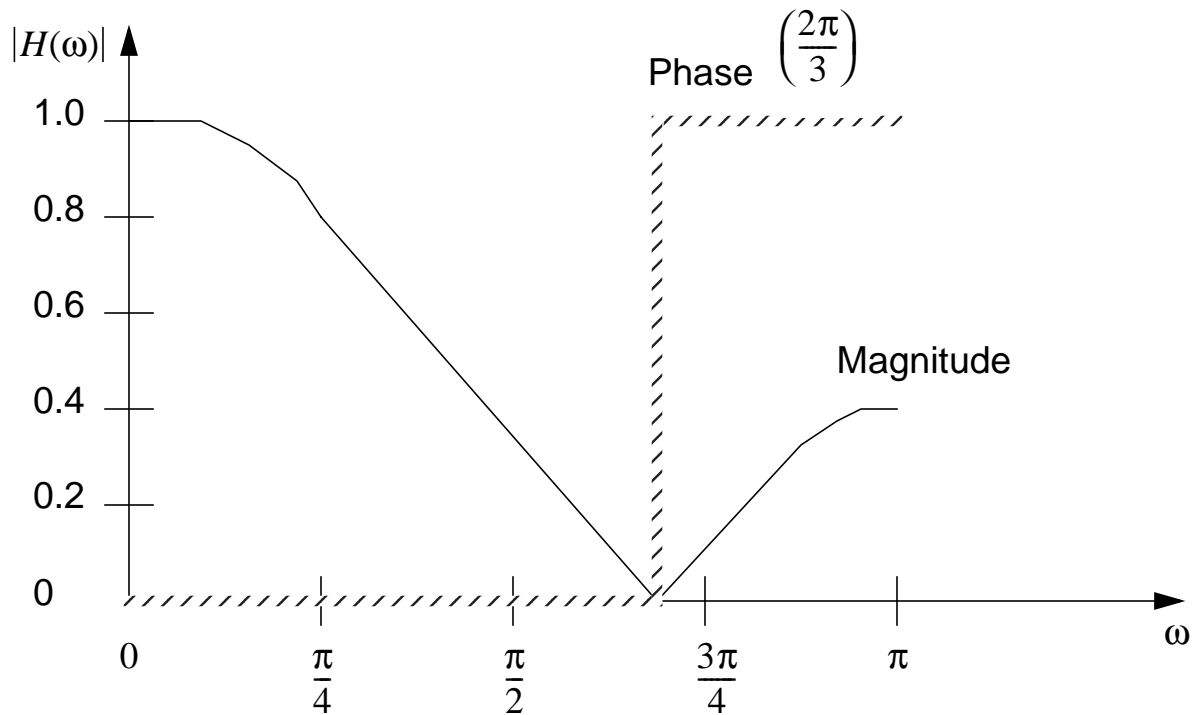
Moving Average Filter (Median Filter):

$$y(n) = \frac{1}{3}[x(n-1) + x(n) + x(n+1)]$$

$$h(n) = \left\{ \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right\}$$

$$H(\omega) = \frac{1}{3}(e^{j\omega} + 1 + e^{-j\omega}) = \frac{1}{3}(1 + 2\cos\omega)$$

$$|H(\omega)| = \frac{1}{3}|(1 + 2\cos\omega)|$$



**Example:**

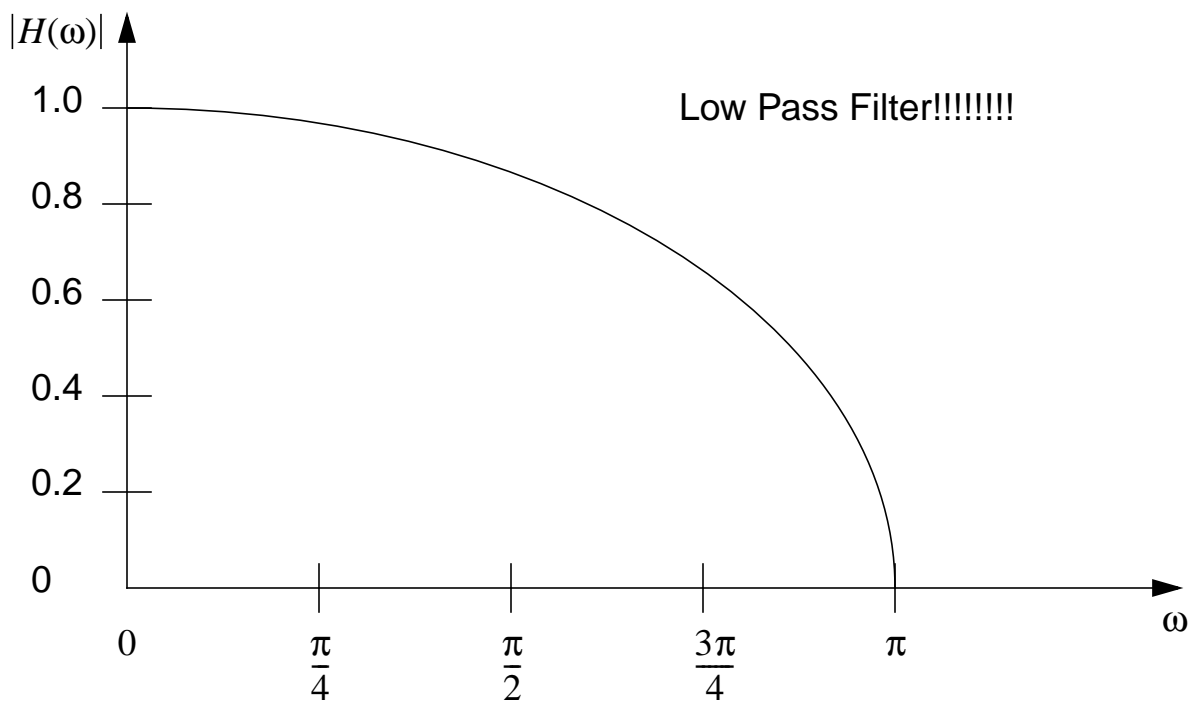
Linear Interpolator:

$$y(n) = \frac{1}{2}[x(n) + x(n-1)]$$

$$h(n) = \left\{ \frac{1}{2}, \frac{1}{2} \right\}$$

$$H(\omega) = \frac{1}{2}(1 + e^{-j\omega})$$

$$\begin{aligned} |H(\omega)| &= \frac{1}{2}|1 + \cos \omega - j \sin \omega| \\ &= \frac{1}{2}\sqrt{(1 + \cos \omega)^2 + (\sin \omega)^2} \\ &= \frac{1}{2}\sqrt{(1 + 2 \cos \omega + (\cos \omega)^2) + (\sin \omega)^2} \\ &= \frac{1}{2}\sqrt{2 + 2 \cos \omega} \\ &= \frac{1}{\sqrt{2}}\sqrt{1 + \cos \omega} \end{aligned}$$



**Example:**

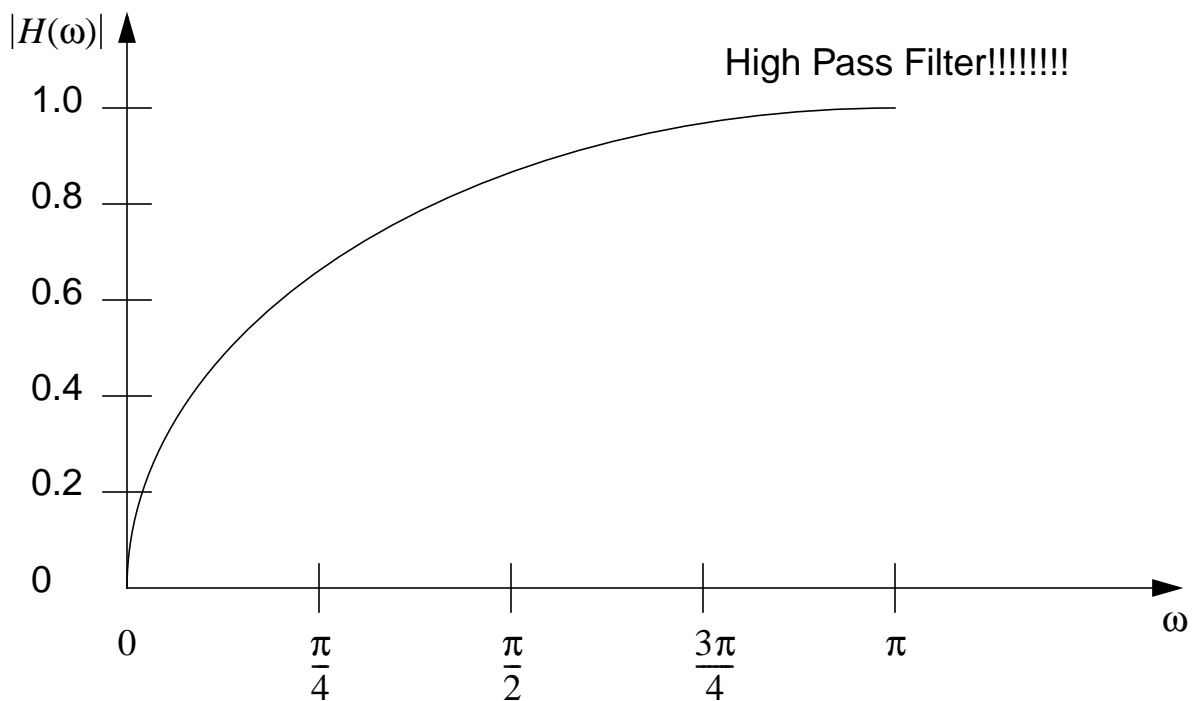
Differentiator:

$$y(n) = \frac{1}{2}[x(n) - x(n-1)]$$

$$h(n) = \left\{ -\frac{1}{2}, \frac{1}{2} \right\}$$

$$H(\omega) = \frac{1}{2}(1 - e^{-j\omega})$$

$$\begin{aligned} |H(\omega)| &= \frac{1}{2}|1 - \cos \omega + j \sin \omega| \\ &= \frac{1}{2}\sqrt{(1 - \cos \omega)^2 + (\sin \omega)^2} \\ &= \frac{1}{2}\sqrt{(1 - 2\cos \omega + (\cos \omega)^2) + (\sin \omega)^2} \\ &= \frac{1}{2}\sqrt{2 - 2\cos \omega} \\ &= \frac{1}{\sqrt{2}}\sqrt{1 - \cos \omega} \end{aligned}$$



## Steady-State and Transient Response to Sinusoidal Input Signals

Consider:

$$y(n) = ay(n-1) + x(n)$$

Its response is given by:

$$y(n) = a^{n+1}y(-1) + \sum_{k=0}^n a^k x(n-k) \quad n \geq 0$$

Suppose:

$$x(n) = Ae^{j\omega n} \quad n \geq 0$$

It can be shown:

$$y(n) = a^{n+1}y(-1) - \frac{Aa^{n+1}e^{-j\omega(n+1)}}{1-ae^{-j\omega}}e^{j\omega n} + \frac{A}{1-ae^{-j\omega}}e^{j\omega n} \quad n \geq 0$$

Note that:

$$\begin{aligned} y_{ss}(n) &= \lim_{n \rightarrow \infty} y(n) = \frac{A}{1-ae^{-j\omega}}e^{j\omega n} \\ &= AH(\omega)e^{j\omega n} \end{aligned}$$

and,

$$y_{tr}(n) = a^{n+1}y(-1) - \frac{Aa^{n+1}e^{-j\omega(n+1)}}{1-ae^{-j\omega}}e^{j\omega n} \quad n \geq 0$$

## Steady-State Response to Periodic Input Signals:

$$x(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N}$$

$$y_k(n) = c_k H\left(\frac{2\pi}{N}k\right) e^{j2\pi kn/N}$$

$$y(n) = \sum_{k=0}^{N-1} c_k H\left(\frac{2\pi}{N}k\right) e^{j2\pi kn/N}$$

Response is periodic with same period as input. The amplitude and phases of the output sinusoidal components can be changed.

# Response to Aperiodic Input Signals:

