Properties of the Fourier Transform for Discrete-Time Signals

The Fourier transform for aperiodic finite-energy discrete-time signals is defined as:

$$X(\omega) \equiv F[x(n)] = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}.$$

The inverse transform is defined as:

$$x(n) \equiv F^{-1}\{X(\omega)\} = \frac{1}{2\pi}\int_{2\pi} X(\omega)e^{j\omega n}d\omega$$

Notes:

- $X(\omega)$ is periodic with period 2π
- Usually we plot the spectrum for the interval $[-\pi,\pi]$ ($[0,\pi]$ or $[0,f_s/2]$ for real signals).

We refer to $x(n)$ and $X(\omega)$ as a Fourier transform pair:

$$x(n) \overset{F}{\leftrightarrow} X(\omega)$$

Symmetry Properties of the Fourier Transform

Suppose that $x(n)$ and $X(\omega)$ are complex-valued. They can be expressed in rectangular form:

$$x(n) = x_R(n) + jx_I(n)$$

$$X(\omega) = X_R(\omega) + jX_I(\omega)$$

By noting that $e^{j\omega} = \cos\omega + j\sin\omega$, we can show:

$$X_R(\omega) = \sum_{n=-\infty}^{\infty} [x_R(n)\cos\omega n + x_I(n)\sin\omega n]$$

$$X_I(\omega) = -\sum_{n=-\infty}^{\infty} [x_R(n)\sin\omega n - x_I(n)\cos\omega n]$$

and,

$$x_R(n) = \frac{1}{2\pi}\int_{2\pi} [X_R(\omega)\cos\omega n - X_I(\omega)\sin\omega n]d\omega$$

$$x_I(n) = \frac{1}{2\pi}\int_{2\pi} [X_R(\omega)\sin(\omega n) + X_I(\omega)\cos\omega n]d\omega$$

These equations form the basis for our exploration of symmetry properties.

Real Signals:

Real/Imaginary Symmetry:

$$X_R(\omega) = \sum_{n=-\infty}^{\infty} x(n)\cos\omega n$$

$$X_I(\omega) = -\sum_{n=-\infty}^{\infty} x(n)\sin\omega n$$

$$X_R(-\omega) = X_R(\omega) \qquad (even)$$

$$X_I(-\omega) = -X_I(\omega) \qquad (odd)$$

Magnitude/Phase Symmetry:

$$|X(\omega)| = |X(-\omega)| \qquad (even)$$

$$\angle X(-\omega) = -\angle X(\omega) \qquad (odd)$$

Real and Even:

$$X_R(\omega) = x(0) + 2\sum_{n=1}^{\infty} x(n)\cos\omega n$$

$$X_I(\omega) = 0$$

Real and Odd:

$$X_R(\omega) = 0$$

$$X_I(\omega) = -2\sum_{n=1}^{\infty} x(n)\sin\omega n$$

Even/Odd Decomposition:

A complex signal can be decomposed into:

$$x(n) = x_e(n) + x_o(n)$$

where:

$$x_e(n) = \frac{1}{2}[x(n) + x^*(n)]$$

$$x_o(n) = \frac{1}{2}[x(n) - x^*(n)]$$

Example:

$$x(n) = \begin{cases} A, & -M \le n \le M \\ 0, & elsewhere \end{cases}$$

$x(n)$ is real and even:

$$X(\omega) = X_R(\omega) = A\left(1 + 2\sum_{n=1}^{\infty} \cos \omega n\right)$$

$$|X(\omega)| = \left| A \frac{\sin\left(M + \frac{1}{2}\right)\omega}{\sin(\omega/2)} \right|$$

$$\angle X(\omega) = \left(\begin{array}{ll} 0, & \text{if } X(\omega) > 0 \\ \pi, & \text{if } X(\omega) < 0 \end{array} \right.$$

Linearity:

$$x_1(n) \xrightarrow{F} X_1(\omega)$$

$$x_2(n) \xrightarrow{F} X_2(\omega)$$

$$a_1 x_1(n) + a_2 x_2(n) \xrightarrow{F} a_1 X_1(\omega) + a_2 X_2(\omega)$$

Time Shifting:

$$x(n) \xrightarrow{F} X(\omega)$$

$$x(n-k) \xrightarrow{F} e^{-j\omega k} X(\omega)$$

Convolution:

$$x_1(n) \xrightarrow{F} X_1(\omega)$$

$$x_2(n) \xrightarrow{F} X_2(\omega)$$

$$x(n) = x_1(n) \otimes x_2(n) \xrightarrow{F} X(\omega) = X_1(\omega) X_2(\omega)$$

Correlation:

$$x_1(n) \xrightarrow{F} X_1(\omega)$$

$$x_2(n) \xrightarrow{F} X_2(\omega)$$

$$r_{x_1 x_2}(m) \xrightarrow{F} S_{x_1 x_2}(\omega) = X_1(\omega) X_2(-\omega)$$

Weiner—Khintchine Theorem (for real $x(n)$)**:**

$$(r_{xx}(m) = x(n) \otimes x(n)) \overset{F}{\rightarrow} S_{xx}(\omega)$$

Frequency Shifting:

$$x(n) \overset{F}{\rightarrow} X(\omega)$$

$$e^{j\omega_0 n} x(n) \overset{F}{\rightarrow} X(\omega - \omega_0)$$

Modulation:

$$x(n) \overset{F}{\rightarrow} X(\omega)$$

$$x(n) \cos \omega_0 n \overset{F}{\rightarrow} \frac{1}{2} X(\omega + \omega_0) + \frac{1}{2} X(\omega - \omega_0)$$

Parseval's Theorem:

$$x_1(n) \overset{F}{\rightarrow} X_1(\omega)$$

$$x_2(n) \overset{F}{\rightarrow} X_2(\omega)$$

$$\sum_{n=-\infty}^{\infty} x_1(n) x_2^*(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(\omega) X_2^*(\omega) d\omega$$

Multiplication (Windowing):

$$x_1(n) \xrightarrow{F} X_1(\omega)$$

$$x_2(n) \xrightarrow{F} X_2(\omega)$$

$$x_3(n) \equiv x_1(n)x_2(n) \xrightarrow{F} X_3(\omega) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\lambda)X_2(\omega-\lambda)d\omega$$

Differentiation in the Frequency Domain:

$$x(n) \xrightarrow{F} X(\omega)$$

$$nx(n) \xrightarrow{F} j\frac{d}{d\omega}X(\omega)$$