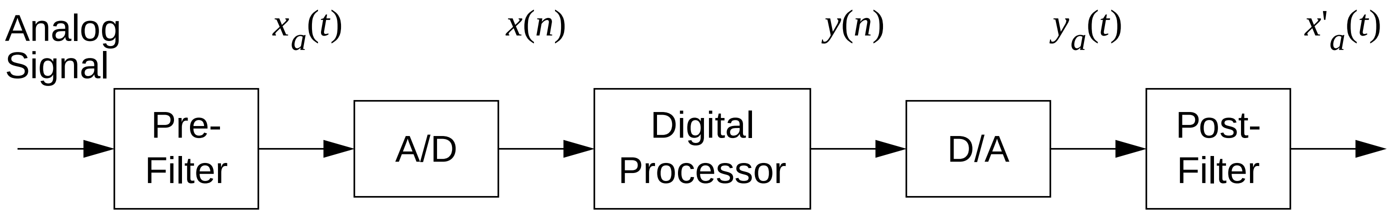
## The Sampling Theorem

Recall a discrete-time signal is given by:

$$(x(n) = x_a(nT)), \qquad -\infty < n < \infty$$




If $x_a(t)$ is an aperiodic signal with finite energy, its spectrum is given by:

$$X_a(F) = \int_{-\infty}^{\infty} x_a(t)e^{-j2\pi Ft}dt$$

The signal can be recovered from the inverse Fourier transform:

$$x_a(t) = \int_{-\infty}^{\infty} X_a(F)e^{j2\pi Ft}dF$$

The spectrum of the discrete-time signal is given by:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

or, equivalently,

$$X(f) = \sum_{n=-\infty}^{\infty} x(n)e^{-j2\pi fn}$$

The signal can be recovered from its spectrum:

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{j\omega n}d\omega$$

$$= \int_{-1/2}^{1/2} X(f)e^{j2\pi fn}df$$

Recall that $t = nT = \frac{n}{F_s}$. This allows us to write the inverse transform as:

$$x(n) \equiv x_a(nT) = \int_{-\infty}^{\infty} X_a(F) e^{j2\pi n(F/F_s)} dF$$

From this, we can conclude that

$$\int_{-1/2}^{1/2} X(f) e^{j2\pi fn} df = \int_{-\infty}^{\infty} X_a(F) e^{j2\pi n(F/F_s)} dF$$

We know that $f = \frac{F}{F_s}$. We can make a change of variables and write:

$$\frac{1}{F_s} \int_{-F_s/2}^{F_s/2} X(\frac{F}{F_s}) e^{j2\pi n(F/F_s)} df = \int_{-\infty}^{\infty} X_a(F) e^{j2\pi n(F/F_s)} dF$$

We can express the integral on the right as a sum of integrals:

$$\int_{-\infty}^{\infty} X_a(F) e^{j2\pi F/F_s} dF = \sum_{k=-\infty}^{\infty} \int_{(k-1/2)F_s}^{(k+1/2)F_s} X_a(F) e^{j2\pi n(F/F_s)} dF$$

By interchanging the order of integration and summation, and invoking the periodicity of the complex exponential, we can write:

$$\frac{1}{F_s} \int_{-F_s/2}^{F_s/2} X(\frac{F}{F_s}) e^{j2\pi n(F/F_s)} df = \int_{-F_s/2}^{F_s/2} \left[ \sum_{k=-\infty}^{\infty} X_a(F - kF_s) \right] e^{j2\pi n(F/F_s)} dF$$

By equating terms inside the integral, we have:

$$X(\frac{F}{F_s}) = F_s \sum_{k=-\infty}^{\infty} X_a(F - kF_s)$$

What does this imply about the spectrum of the sampled signal?

Aliasing is defined as the distortion that is produced by sampling a signal below its Nyquist rate.

We can recover $x(t)$, if there was no aliasing, as follows:

$$X_a(F) = \begin{cases} \frac{1}{F_s} X\left(\frac{F}{F_s}\right) & |F| \le F_s/2 \\ 0 & |F| > F_s/2 \end{cases}$$

By taking the inverse Fourier transform,

$$X\left(\frac{F}{F_s}\right) = \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi F n/F_s}$$

The inverse transform of $X_a(F)$ is

$$x_a(t) = \int_{-F_s/2}^{F_s/2} X(F) e^{j2\pi n F t} dF$$

From substitution,

$$x_a(t) = \frac{1}{F_s} \int_{-F_s/2}^{F_s/2} \left[ \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi F n/F_s} \right] e^{j2\pi F t} dF$$

or,

$$x_a(t) = \frac{1}{F_s} \sum_{n=-\infty}^{\infty} x(n) \int_{-F_s/2}^{F_s/2} e^{j2\pi F(t - n/F_s)} dF$$

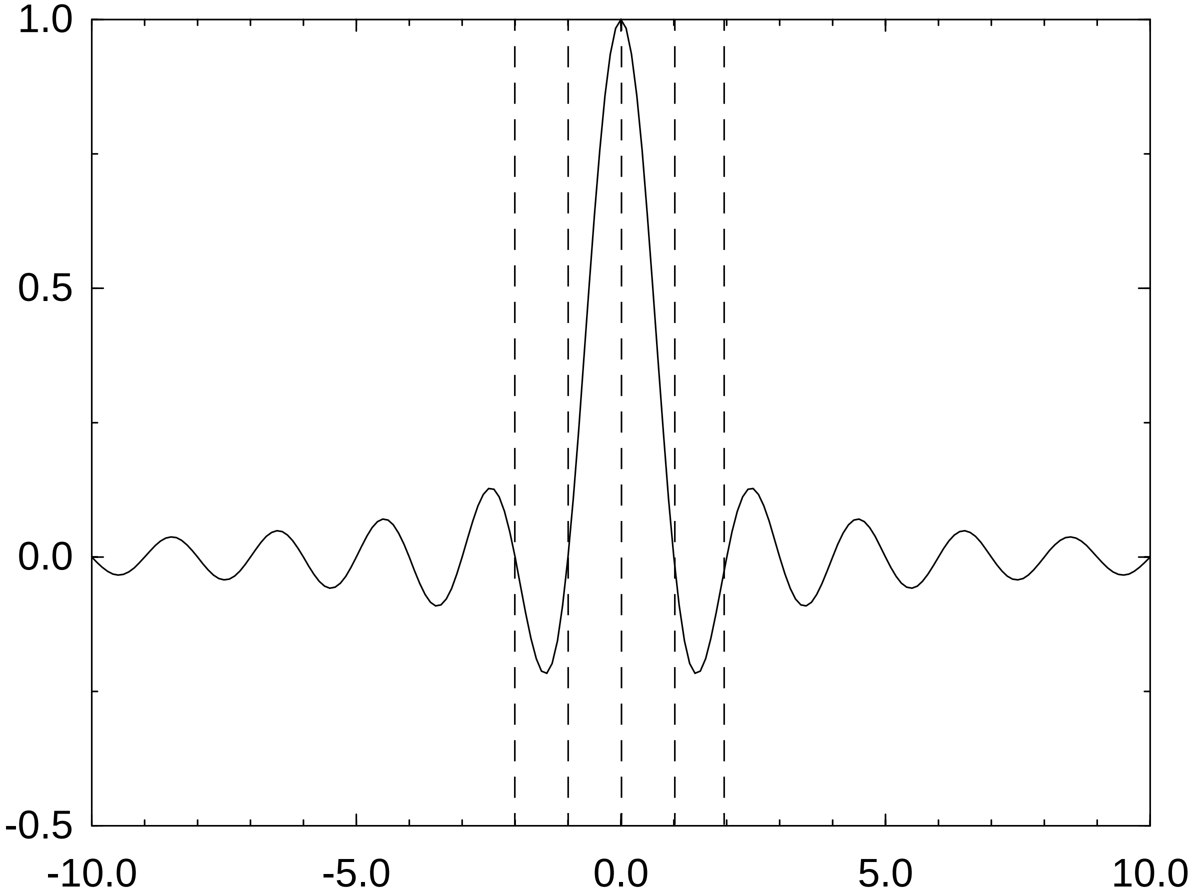
$$= \sum_{n=-\infty}^{\infty} x_a(nT) \frac{\sin((\pi/T)(t-nT))}{(\pi/T)(t-nT)}$$

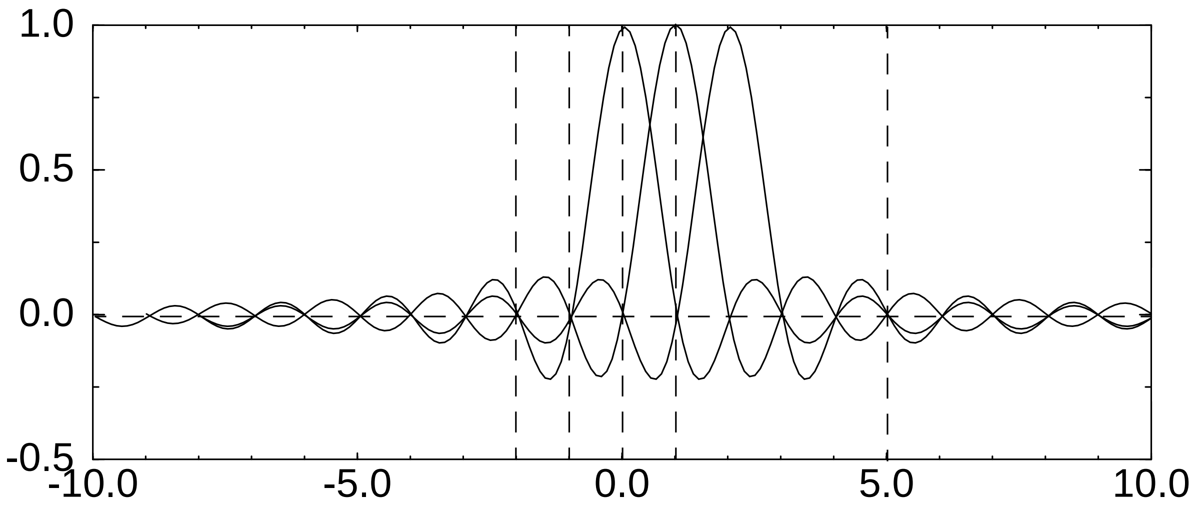
Note that at the original sample instances, the analog signal is equal to the value of the original signal because the sinc functions go to zero. At times between the sample instances, the signal is the weighted sum of shifted sinc functions (see Fig. 6.3).

The above equation forms the basis for the sampling theorem.

# Sin(x)/x Interpolation:

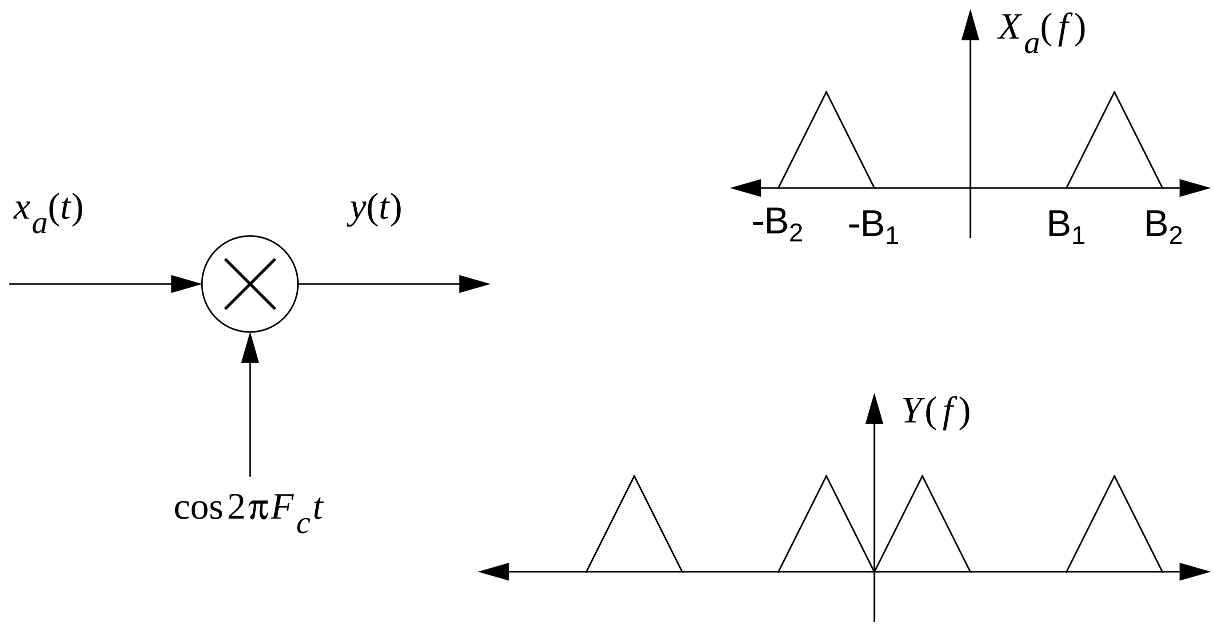
# A Graphical Interpretation of the Sampling Theorem



## The Bandlimited Sampling Theorem (An Intuitive Approach)

Consider the following system:




We can sample a bandpass signal at a frequency lower than its "Nyquist rate" by converting it to a lowpass signal.

In general, we suspect we can directly sample the signal, but we to select a sample frequency such that folding does not cause aliasing.

A general guideline is:

$$2B \le F_s \le 4B$$

A more rigorous equation is:

$$F_s = 2B\frac{r'}{r}$$

where

$$r' = \frac{F_c + B/2}{B}$$

and

$r = \lfloor r' \rfloor$ (greatest integer less than or equal to r)

$$F_c = \frac{B_1 + B_2}{2}$$