Design of Lowpass Filters Using Poles and Zeros

- Poles should be placed near the unit circle at points corresponding to low frequencies
- Zeros should be placed on the unit circle for high frequencies

Consider a single pole filter where a is real:

$$H_1(z) = \frac{1-a}{1-az^{-1}}$$

What is the frequency response? Note the unity gain and the "rolloff."

Now consider the addition of a zero on the unit circle:

$$H_2(z) = \frac{1-a}{2}\frac{1+z^{-1}}{1-az^{-1}}$$

What is the frequency response? Is this a good idea? Precision?

Now, consider transforming this lowpass filter to a highpass filter by reflecting the pole-zero locations about the imaginary axis:

$$H_2(z) = \frac{1-a}{2}\frac{1-z^{-1}}{1+az^{-1}}$$

What about the design of a bandpass filter?

Complex-conjugate poles can be used, but we end up with a second-order filter. Why?

Lowpass To Highpass Transformation

If $h_{lp}(n)$ denotes the impulse response of a lowpass filter with frequency response $H_{lp}(\omega)$, a highpass filter can be obtained as:

$$H_{hp}(\omega) = H_{lp}(\omega - \pi)$$

Since a frequency translation of π is equivalent to multiplication by $e^{j\pi n}$ in the time domain, the impulse response $h_{hp}(n)$ is given by:

$$\begin{aligned} h_{hp}(n) &= (e^{j\pi})^n h_{lp}(n) \\ &= (-1)^n h_{lp}(n) \end{aligned}$$

The corresponding difference equation can be described by:

$$y(n) = -\sum_{k=1}^{N} (-1)^k a_k y(n-k) + \sum_{k=0}^{M} (-1)^k b_k x(n-k)$$

Similarly, $h_{lp}(n) = (-1)^n h_{hp}(n)$.

Digital Resonators

A digital resonator is a special two-pole bandpass filter with a pair of complex-conjugate poles located near the unit circle:

$$H(z) = \frac{b_0}{(1 - re^{j\omega_0}z^{-1})(1 - re^{-j\omega_0}z^{-1})}$$
$$= \frac{b_0}{1 - (2r\cos\omega_0)z^{-1} + r^2z^{-2}}$$

How do we set the gain of the filter such that it has unity gain at $\omega = \omega_0$?

The desired normalization factor is:

$$b_0 = (1 - r)\sqrt{1 + r^2 - 2r\cos 2\omega_0}$$

Note that the filter's resonant frequency is:

$$\omega_r = \cos^{-1}\left(\frac{1 + r^2}{2r}\cos\omega_0\right)$$

What happened?

As $r \to 1$, $\omega_r \to \omega_0$, why?

For $r \approx 1$, the bandwidth (point at which response is down 3 dB) can be approximated by $\Delta\omega \approx 2(1 - r)$.

What happens if we add zeros:

$$H(z) = b_0\frac{(1 - z^{-1})(1 + z^{-1})}{1 - (2r\cos\omega_0)z^{-1} + r^2z^{-2}}$$

Notch Filters

A notch filter is a filter that severely attenuates a “single frequency” or a small group of frequencies. For example, in many systems, we try to reject line noise (“60 Hz hum”).

What about:

$$H(z) = b_0(1 - e^{j\omega_0}z^{-1})(1 - e^{-j\omega_0}z^{-1})$$
$$= b_0(1 - 2\cos\omega_0 z^{-1} + z^{-2})$$

We can also add poles:

$$H(z) = b_0 \frac{1 - 2\cos\omega_0 z^{-1} + z^{-2}}{1 - 2r\cos\omega_0 z^{-1} + r^2 z^{-2}}$$

What does this do to the frequency response?

Phase response?

Comb Filters

Consider the system:

$$y(n) = \frac{1}{M+1}\sum_{k=0}^{M} x(n-k)$$

The FIR filter for this system is given by:

$$H(z) = \frac{1}{M+1}\frac{[1-z^{-(M+1)}]}{1-z^{-1}}$$

The frequency response is:

$$H(\omega) = \frac{e^{-j\omega\frac{M}{2}}}{M+1}\frac{\sin\omega\left(\frac{M+1}{2}\right)}{\sin(\omega/2)}$$

Note that the zeros are at $z = e^{j2\pi k/(M+1)}$, $\quad k = 1, 2, \ldots, M$.

Consider:

$$H(z) = \sum_{k=0}^{M} h(k)z^{-k}$$

Let $z = z^L$:

$$H_L(z) = \sum_{k=0}^{M} h(k)z^{-kL}$$

What is the effect?

$$H_L(\omega) = H(L\omega)$$

Consider:

$$H(z) = \frac{1}{M+1}\frac{[1-z^{-L(M+1)}]}{1-z^{-L}}$$

Where are the zeros?

All-Pass Filters

Consider:

$$H(z) = \frac{a_N + a_{N-1}z^{-1} + \ldots + a_1 z^{-N+1} + z^{-N}}{1 + a_1 z^{-1} + \ldots + a_N z^{-N}}$$

$$= \frac{\sum_{k=0}^{N} a_k z^{-N+k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

Let all the filter coefficients be real and $a_0 = 1$. Define:

$$A(z) = \sum_{k=0}^{N} a_k z^{-k}$$

Then,

$$H(z) = z^{-N} \frac{A(z^{-1})}{A(z)}$$

Note that

$$|H(\omega)|^2 = H(z)H(z^{-1})\Big|_{z = e^{j\omega}} = 1$$

The general form of this is:

$$H_{ap}(z) = \prod_{k=1}^{N_R} \frac{z^{-1} - \alpha_k}{1 - \alpha_k z^{-1}} \prod_{k=1}^{N_c} \frac{(z^{-1} - \beta_k)(z^{-1} - \beta_k^*)}{(1 - \beta_k z^{-1})(1 - \beta_k^* z^{-1})}$$