Characteristics of Commonly Used Analog Filters - Butterworth

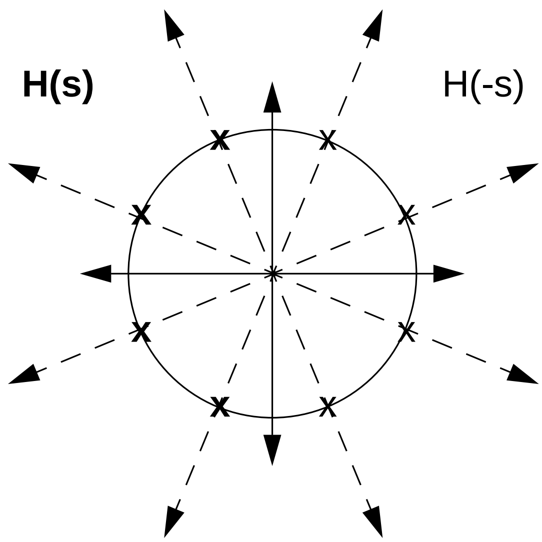
Butterworth filters are maximally flat in the passband and stopband, giving monotonicity priority over transition-band attenuation. These are all-pole filters (in the analog domain) characterized by the frequency response:

$$|H(\Omega)|^2 = \frac{1}{1+(\Omega/\Omega_c)^{2N}} = \frac{1}{1+\varepsilon^2(\Omega/\Omega_p)^{2N}}$$

where N is the order, Ω_c is the cutoff (3 dB) frequency, Ω_p is the passband edge frequency, and $1/(1+\varepsilon^2)$ is the gain at the edge of the passband (at $\Omega = \Omega_p$).

Where are the poles of this filter?

$$s_k = \Omega_c e^{j2\pi/2} e^{j(2k+1)\pi/(2N)} \qquad k = 0, 1, \ldots, N-1$$



We normally design a Butterworth filter by specifying a required attenuation, δ_2, at a given frequency, Ω_s:

$$\frac{1}{1+\varepsilon^2(\Omega/\Omega_p)^{2N}} = \delta_2^2$$

A relationship between the order and the attenuation is:

$$N = \frac{\log[(1/\delta_2^2)-1]}{2\log(\Omega_s/\Omega_c)}$$

Thus, a Butterworth filter is completely specified by N, δ_2, ε, and the ratio Ω_s/Ω_c.

Chebyshev Type I and II Filters

Type I: Equiripple in the passband, monotonic in the stopband

Type II: Monotonic behavior in the passband, equiripple in the stopband

⇨ Maximize attenuation at the expense of ripple

The transfer function for a Type I filter is:

$$|H(\Omega)|^2 = \frac{1}{1 + \varepsilon^2 T_N^2(\Omega/\Omega_p)}$$

where ε is a parameter controlling passband ripple, and $T_N(x)$ is the N-th order Chebyshev polynomial defined as:

$$T_N(x) = \begin{cases} \cos(N\cos^{-1}x), & |x| \le 1 \\ \cosh(N\cosh^{-1}x), & |x| > 1 \end{cases}$$

The Chebyshev polynomials can be generated by the recursive equation:

$$T_{N+1}(x) = 2xT_N(x) - T_{N-1}(x), \qquad N = 1, 2, \ldots$$

where $T_0(x) = 1$ and $T_1(x) = x$. A Chebyshev filter has poles distributed about an ellipse that lies within the unit circle (see Figure 8.38 on page 637).

The transfer function for a Type II filter contains poles and zeroes (located on the imaginary axis):

$$|H(\Omega)|^2 = \frac{1}{1 + \varepsilon^2[T_N^2(\Omega/\Omega_p)/T_N^2(\Omega_s/\Omega)]}$$

A Chebyshev filter is completely specified by N, δ_2, ε, and the ratio Ω_s/Ω_p. We can derive a design equation for the order in terms of these parameters:

$$N = \frac{\log\left[\left(\sqrt{1-\delta_2^2} + \sqrt{1-\delta_2^2(1+\varepsilon^2)}\right)/(\varepsilon\delta_2)\right]}{\log\left[(\Omega_s/\Omega_p) + \sqrt{(\Omega_s/\Omega_p)^2 - 1}\right]}$$

where $\delta_2 = 1/\left(\sqrt{1+\delta^2}\right)$.

$$= \frac{\cosh^{-1}(\delta/\varepsilon)}{\cosh^{-1}(\Omega_s/\Omega_p)}$$

Elliptic (Cauer) Filters

Equiripple in both the stopband and passband. The transfer function is given by:

$$|H(\Omega)|^2 = \frac{1}{1+\varepsilon^2 U_N(\Omega/\Omega_p)}$$

where $U_N(x)$ is the Jacobian elliptic function of order N, and ε is a parameter related to the passband ripple. The zeros of this filter lie on the imaginary axis. These filters are the most efficient, because they distribute ripple equally in the stopband and passband. This implies we get the maximum attenuation for a given order. What about phase?

A design equation for the filter order is given by:

$$K = \frac{K(\Omega_p/\Omega_s)K(\sqrt{1-(\varepsilon^2/\delta^2)})}{K(\varepsilon/\delta)K\left(\sqrt{1-(\Omega_p/\Omega_s)^2}\right)}$$

where $K(x)$ is the complete elliptic integral of the first kind, defined as:

$$K(x) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-x^2\sin^2\theta}}$$

and $\delta_2 = 1/\left(\sqrt{1+\delta^2}\right)$. The passband ripple is $10\log_{10}(1+\varepsilon^2)$.

These filters are very popular for consumer applications because of their efficiency. There are many third-party software packages that support the automated design of these filters (very hard to design by hand!).

Bessel Filters

Bessel filters are all-pole filters with a system function given by:

$$H(s) = \frac{1}{B_N(s)}$$

where $B_N(s)$ is the N-th order Bessel polynomial. These polynomials are of the form:

$$B_N(s) = \sum_{k=0}^{N} a_k s^k$$

where the coefficients $\{a_k\}$ are given as

$$a_k = \frac{(2N-k)!}{2^{N-k}k!(N-k)!} \qquad k = 0, 1, \ldots, N$$

These can also be computed recursively:

$$B_N(s) = (2N-1)B_{N-1}(s) + s^2 B_{N-2}(s)$$

with $B_0(s) = 1$ and $B_1(s) = s + 1$ as initial conditions.

Bessel filters have a linear phase response in the passband in the analog domain — why might this be useful.

However, when the filter is transformed into the discrete domain, the linear phase characteristic is lost.

Finally, consider **Picone** filters:

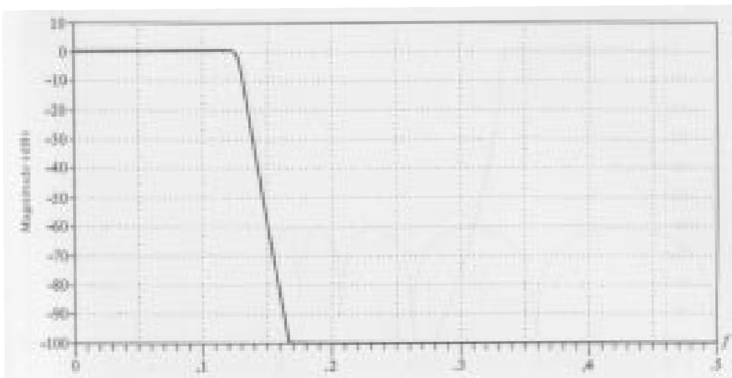
$$H(s) = \frac{1}{P_N(s)}$$

what would it take to convince you that such a filter was worthwhile?

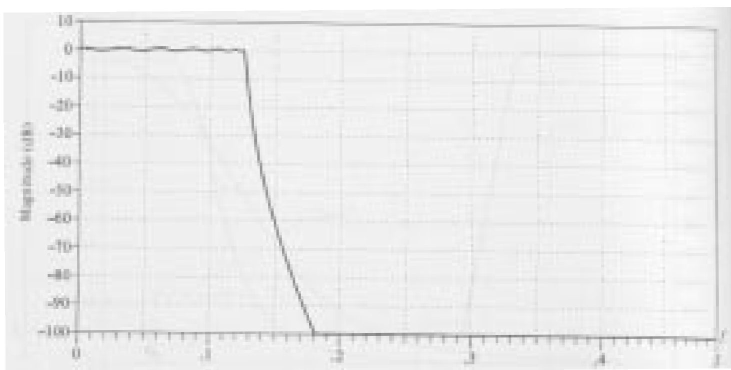
A Comparison of Digital Filter Designs

Specification: max. ripple of $1/2$ dB in the passband; passband edge freq.: $\omega_p = 0.25\pi$

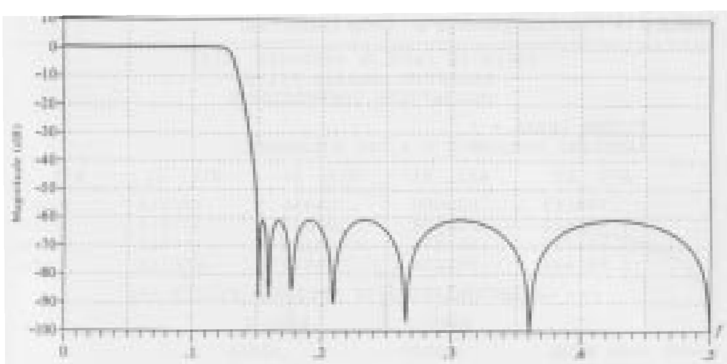
60 dB atten. in the stopband; stopband edge freq.: $\omega_s = 0.30\pi$



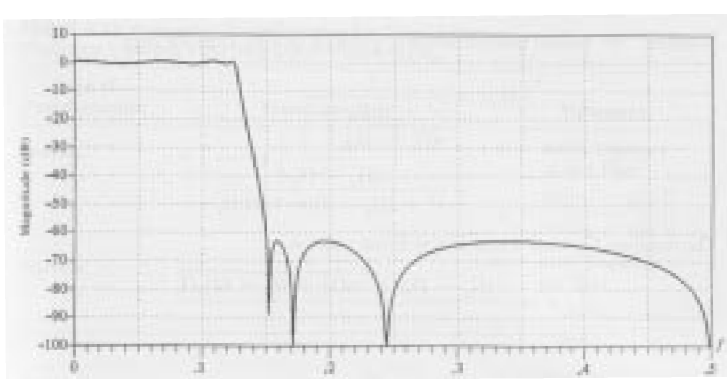
Butterworth
N=39



Chebyshev Type 1
N=13



Chebyshev Type II
N=13



Elliptic
N=7