Locations of the Zeros of a Linear Phase FIlter

An FIR filter can be described by a difference equation:

$$y(n) = \sum_{k=0}^{M-1} h(k)x(n-k) \quad \text{or,} \quad H(z) = \sum_{k=0}^{M-1} h(k)z^{-k}$$

If the filter is a linear phase filter, its impulse response MUST satisfy the constraint:

$$h(n) = \pm h(M-1-n) \qquad n = 0, 1, \ldots, M-1$$

"+" corresponds to the symmetry case, "-" corresponds to antisymmetry. We can compactly represent the frequency response as:

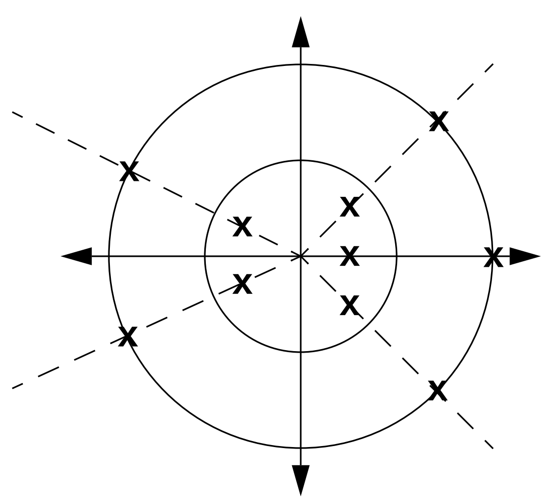
$$m \text{ odd: } H(z) = z^{-(M-1)/2}\left\{h\left(\frac{M-1}{2}\right) + \sum_{n=0}^{(M-3)/2} h(n)[z^{(M-1-2n)/2} \pm z^{-(M-1-2n)/2}]\right\}$$

$$m \text{ even: } H(z) = z^{-(M-1)/2}\left\{\sum_{n=0}^{(M/2)-1} h(n)[z^{(M-1-2n)/2} \pm z^{-(M-1-2n)/2}]\right\}$$

If we substitute z^{-1} for z, and multiply both sides by $z^{-(M-1)}$, we obtain:

$$z^{-(M-1)}H(z^{-1}) = \pm H(z)$$

This implies that the roots of $H(z)$ occur in reciprocal pairs, and conjugate pairs if $h(n)$ has real coefficients:



Design of Linear-Phase FIR Filters Using Windows

Suppose we want to design a linear phase lowpass FIR filter:

$$H_d(\omega) = \begin{cases} 1e^{-j\omega(M-1)/2} & 0 \le \omega \le \omega_c \\ 0 & otherwise \end{cases}$$

We can compute $h_d(n)$ using the inverse transform:

$$h_d(n) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega\left(n-\frac{M-1}{2}\right)}\, d\omega$$

$$= \frac{\sin\omega_c\left(n-\frac{M-1}{2}\right)}{\pi\left(n-\frac{M-1}{2}\right)}$$

Clearly, $h_d(n)$ is noncausal and infinite in duration. We can truncate using a window:

$$h(n) = h_d(n)w(n)$$

What are the drawbacks of this approach?

Can we generalize this?

Hamming window: $0.54 - 0.46\cos\left(\frac{2\pi n}{M-1}\right)$

Hanning window: $\frac{1}{2}\left(1-\cos\left(\frac{2\pi n}{M-1}\right)\right)$

Kaiser window: $\dfrac{I_0\left[\alpha\sqrt{\left(\frac{M-1}{2}\right)^2-\left(n-\frac{M-1}{2}\right)^2}\right]}{I_0\left[\alpha\left(\frac{M-1}{2}\right)\right]}$

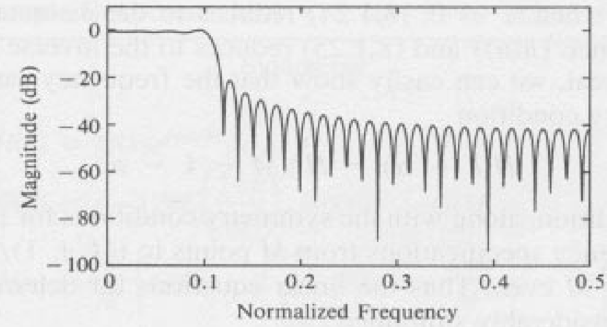


FIGURE 8.8 Lowpass FIR filter designed with rectangular window ($M = 61$).

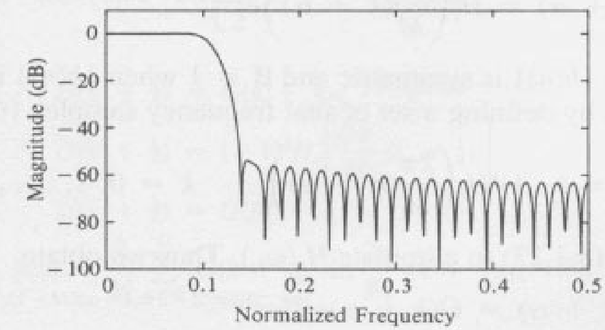


FIGURE 8.9 Lowpass FIR filter designed with Hamming window ($M = 61$).

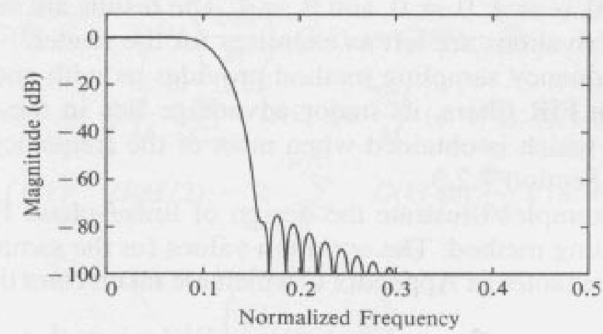


FIGURE 8.10 Lowpass FIR filter designed with Blackman window ($M = 61$).

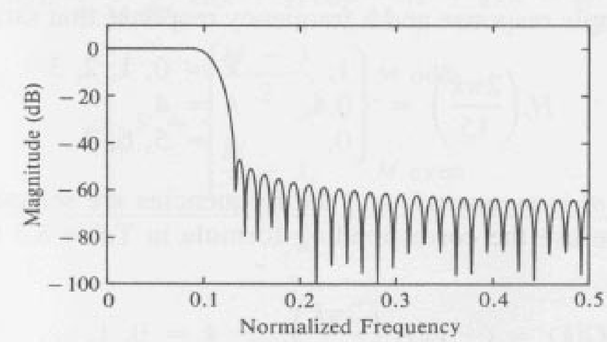


FIGURE 8.11 Lowpass FIR filter designed with $\alpha = 4$ Kaiser window ($M = 61$).

Design of Linear-Phase FIR Filters By Frequency Sampling Methods

Let $H_d(\omega)$ be specified by an equispaced set of samples:

$$H_d(k+\alpha) \equiv H_d(\omega)\Big|_{\omega = \frac{2\pi}{M}(k+\alpha)} \qquad k = 0, 1, \ldots, \frac{M-1}{2} \qquad (M\ odd)$$

$$k = 0, 1, \ldots, \frac{M}{2} - 1 \qquad (M\ even)$$

Then, we can compute the filter impulse response from the inverse transform:

$$h(n) = \frac{1}{M}\sum_{k=0}^{M-1} H_d(k+\alpha)e^{j2\pi(k+\alpha)n/M}$$

Since $\{h(n)\}$ are real, we can show:

$$H(k+\alpha) = H^*(M-k-\alpha)$$

By defining a set of real frequency samples, $\{G(k+\alpha)\}$, we can simplify this design approach as follows:

$$G(k+\alpha) = (-1)^k \left| H_d\left(\frac{2\pi}{M}(k+\alpha)\right)\right| \qquad k = 0, 1, \ldots, M-1,$$

we can show:

$$H(k+\alpha) = G(k+\alpha)e^{j\pi k}e^{j[\beta\pi/2 - 2\pi(k+\alpha)(M-1)/2M]}$$

This covers four cases (see Table 8.3):

- symmetric $\beta = 0$)/antisymmetric ($\beta = 1$)

- $\alpha = 0/\alpha = 1$

Why is this still not a useful design methodology?

Design of Optimum Equiripple Linear-Phase FIR Filters

Consider the problem:

$$1 - \delta_1 \le H_r(\omega) \le 1 + \delta_1 \qquad |\omega| \le \omega_p$$
$$-\delta_2 \le H_r(\omega) \le \delta_2 \qquad |\omega| \ge \omega_s$$

Suppose we constrain our choices for possible filters to a linear phase filter where $h(n) = h(M-1-n)$, and M is odd. Then,

$$H_r(\omega) = h\left(\frac{M-1}{2}\right) + 2\sum_{n=0}^{(M-3)/2} h(n)\cos\left(\omega\left(\frac{M-1}{2} - n\right)\right)$$

Let $k = \frac{M-1}{2} - n$, and:

$$a(k) = \begin{cases} h\left(\frac{M-1}{2}\right) & k = 0 \\ 2h\left(\frac{M-1}{2} - k\right) & k = 1, 2, \ldots, \frac{M-1}{2} \end{cases}$$

Using these definitions,

$$H_r(\omega) = \sum_{k=0}^{(M-1)/2} a(k)\cos\omega k$$

Our strategy is to solve for $\{a(k)\}$ from $H_r(\omega)$, and then $h(n)$ from $\{a(k)\}$.

Let us add an optimization component to the problem (let the user decide what aspects of the design are important):

$$H_r(\omega) = Q(\omega)P(\omega)$$

where $Q(\omega) = 1$ and $P(\omega) = \sum_{k=0}^{L} a(k)\cos\omega k$

Define a real-valued weighting function:

$$H_{dr}(\omega) = \left\{\begin{matrix} 1 & |\omega| \le \omega_p \\ 0 & |\omega| \ge \omega_s \end{matrix}\right\} \quad \text{and} \quad W(\omega) = \begin{cases} \delta_2/\delta_1 & |\omega| \le \omega_p \\ 0 & |\omega| \ge \omega_s \end{cases}$$

The weighted error can be expressed as:

$$E(\omega) = W(\omega)[H_{dr}(\omega) - H_r(\omega)]$$
$$= W(\omega)Q(\omega)[H_{dr}(\omega)/Q(\omega) - P(\omega)]$$

or,

$$E(\omega) = \hat{W}(\omega)[\hat{H}_{dr}(\omega) - P(\omega)]$$

where,

$$\hat{W}(\omega) = W(\omega)Q(\omega)$$
$$\hat{H}_{dr}(\omega) = H_{dr}(\omega)/Q(\omega)$$

We would like a procedure to minimize $E(\omega)$:

Alternation Theorem: *Let S be a compact subset of the interval $[0, \pi]$. A necessary and sufficient condition for:*

$$P(\omega) = \sum_{k=0}^{L} a(k)\cos\omega k$$

to be the unique, best-weighted Chebyshev approximation to $\hat{H}_{dr}(\omega)$ in S is that the error function $E(\omega)$ exhibit at least $L + 2$ extremal frequencies in S. That is, there must exist at least $L + 2$ frequencies $\{\omega_i\}$ in S such that:

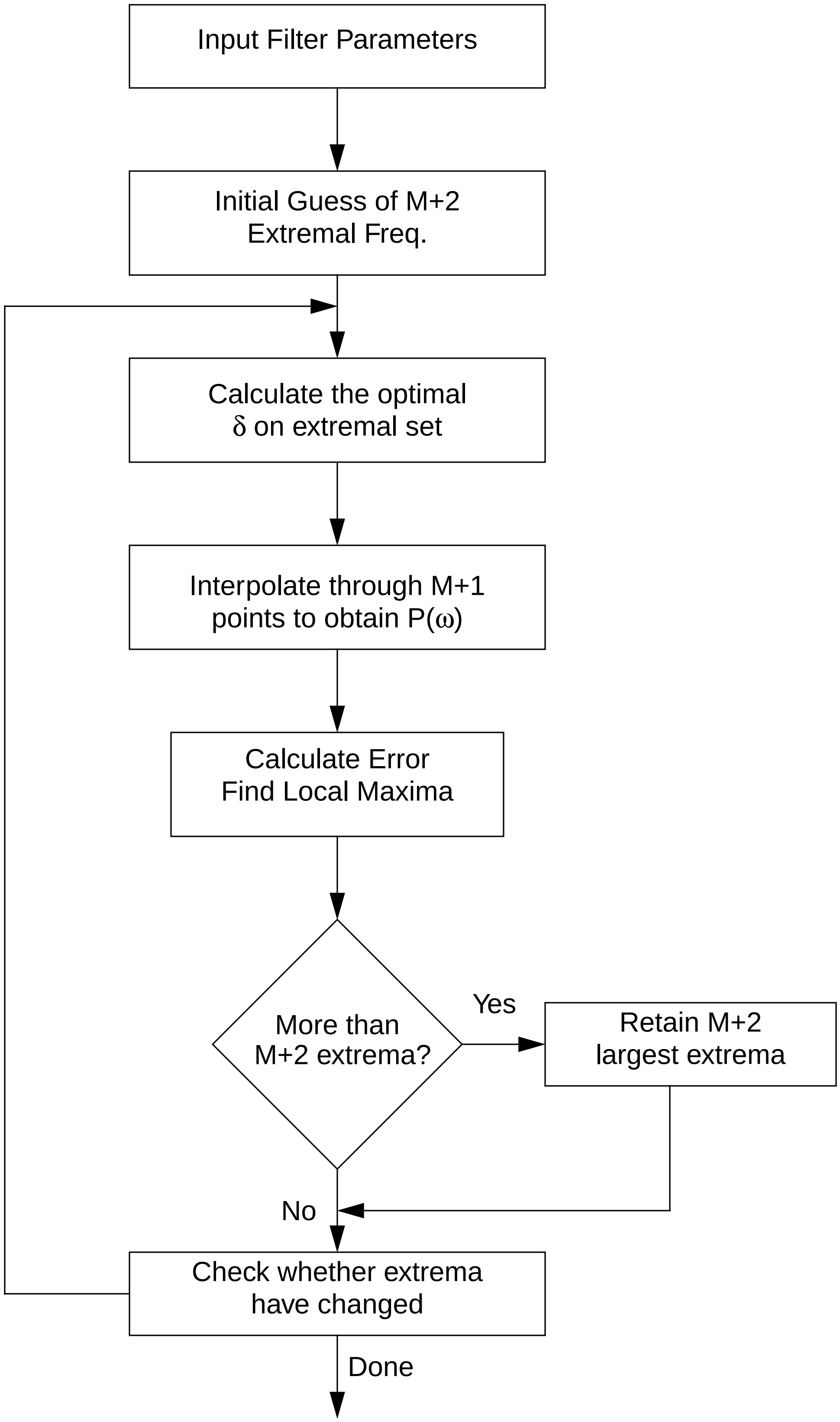
$$\omega_1 \le \omega_2 \le \ldots \le \omega_{L+2}$$

$$E(\omega_i) = -E(\omega_{i+1})$$

$$|E(\omega_i)| = \max_{\omega \in S} |E(\omega)| \qquad i = 1, 2, \ldots, L + 2$$

$E(\omega)$ alternates in sign between a maximum and minimum, hence the theorem is called the alternation theorem. Several procedures exist to find $P(\omega)$. The most famous is the Remez exchange algorithm:

An Overview of the Remez Exchange Algorithm



Parameters Of The Parks-McLellan Program

NFILT: The filter length, denoted above as M.

JYTPE: The type of filter:

JTYPE=1 results in a multiple passband/stopband filter.
JTYPE=2 results in a differentiator.
JTYPE=1 results in a Hilbert transformer.

NBANDS: The number of frequency bands (typically ranges from 2 for a lowpass to a software-dependent maximum for a multiple-band filter).

LGRID: The grid density for interpolating the error function (usually 16 by default).

EDGE: Lower and upper cutoff frequencies of the bands.

FX: Desired frequency response of each band (band gain).

WTX: Weight function in each band.

This algorithm can be found embedded in many tools, including Matlab.

What is wrong with this approach?