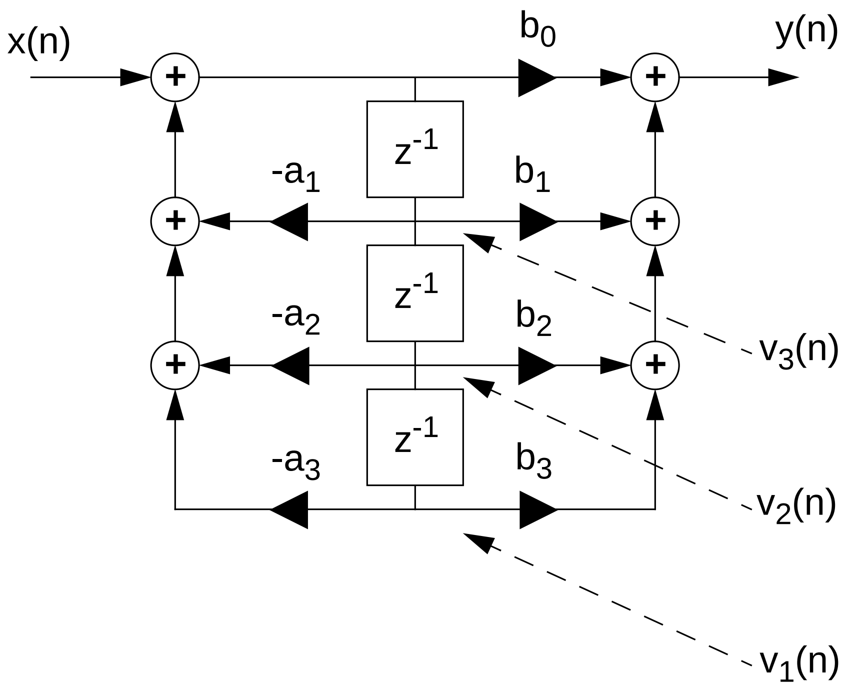
State-Space Descriptions of Systems

Recall the Direct-Form II realization of a discrete-time system:

$$y(n) = -\sum_{k=1}^{3} a_k y(n-k) + \sum_{k=0}^{3} b_k x(n-k)$$



Recall we can write this difference equation as:

$$v(n) = -\sum_{k=1}^{3} a_k v(n-k) + x(n) \qquad y(n) = \sum_{k=0}^{3} b_k v(n-k)$$

Using $v(n)$ as an intermediate set of variables, we can write:

$$v_1(n+1) = v_2(n)$$

$$v_2(n+1) = v_3(n)$$

$$v_3(n+1) = -a_3 v_1(n) - a_2 v_2(n) - a_1 v_3(n) + x(n)$$

and,

$$\begin{aligned} y(n) &= b_0 v_3(n+1) + b_3 v_1(n) + b_2 v_2(n) + b_1 v_3(n) \\ &= (b_3 - b_0 a_3) v_1(n) + (b_2 - b_0 a_2) v_2(n) + (b_1 - b_0 a_1) v_3(n) + b_0 x(n) \end{aligned}$$

Matrix Forms For State-Space Descriptions

Why a matrix formulation?

$$\begin{bmatrix} v_1(n+1) \\ v_2(n+1) \\ v_3(n+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} v_1(n) \\ v_2(n) \\ v_3(n) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} x(n)$$

$$y(n) = \begin{bmatrix} (b_3 - b_0 a_3) & (b_2 - b_0 a_2) & (b_1 - b_0 a_1) \end{bmatrix} \begin{bmatrix} v_1(n) \\ v_2(n) \\ v_3(n) \end{bmatrix} + b_0 x(n)$$

Is the ONLY formulation for this system?
How does this formulation compare to the analog version?

In the general case, we have:

$$\bar{v}(n+1) = \boldsymbol{F}\bar{v}(n) + \bar{q}x(n)$$

$$y(n) = \bar{g}^t \bar{v}(n) + d x(n)$$

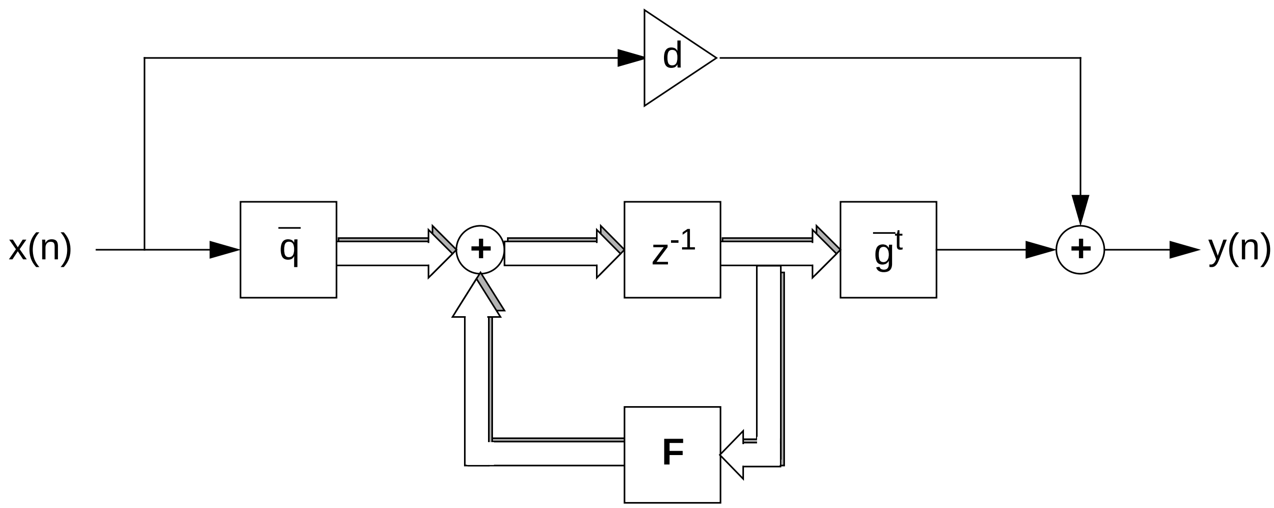
$\{v_i(n)\}$ are called state variables.

$$\boldsymbol{F} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & 1 \\ -a_N & -a_{N-1} & \dots & -a_2 & -a_1 \end{bmatrix} \qquad \bar{q} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \dots \\ 1 \end{bmatrix}$$

$$\bar{g} = \begin{bmatrix} b_N - b_0 a_N \\ b_{N-1} - b_0 a_{N-1} \\ \dots \\ b_2 - b_0 a_2 \\ b_1 - b_0 a_1 \end{bmatrix} \qquad d = b_0$$

$\boldsymbol{F}$, $\bar{q}$, $\bar{g}$, and d are CONSTANTS and define the properties of the system.

General State-Space Descriptions



- This multichannel signal flow graph describes ALL linear time-invariant systems. Why is such a model useful?
- What are possible alternate choices for the state variables?
- Note than an N-dimensional differential (or difference) equation can be rewritten as N first-order equations.

Solutions of the State-Space Equations

For the initial condition, $\bar{v}(n_0)$,

$$\bar{v}(n) = \boldsymbol{F}^{n-n_0} v(n_0) + \sum_{k=n_0}^{n-1} \boldsymbol{F}^{n-1-k} \bar{q} x(k)$$

where, $\boldsymbol{F}^0$ is an identity matrix, $\boldsymbol{F}^2$ is the product of $\boldsymbol{F}$ and $\boldsymbol{F}$, and $\Phi(i-j) \equiv \boldsymbol{F}^{i-j}$. With these definitions, the output is given by:

$$y(n) = \bar{g}^t \Phi(n-n_0) \bar{v}(n_0) + \sum_{n=n_0}^{n-1} \bar{g}^t \Phi(n-1-k) \bar{q} x(k) + d x(n)$$

Solutions of the State-Space Equations (cont.)

For zero-input:

$$y_{zi}(n) = \bar{g}^t \Phi(n - n_0)\bar{v}(n_0)$$

this is the response due to the initial conditions.

The zero-state response (no initial conditions) is:

$$y_{zs}(n) = \sum_{n = n_0}^{n-1} \bar{g}^t \Phi(n - 1 - k)\bar{q}x(k) + dx(n)$$

Note that the total response is the sum of these two.

Relationships Between I/O and State-Space Descriptions

Let $\hat{\bar{v}}(n) = \boldsymbol{P}\bar{v}(n)$ and $\bar{v}(n) = \boldsymbol{P}^{-1}\hat{\bar{v}}(n)$. By substituting into the state-space equations, we can derive an alternate formulation:

$$\boldsymbol{P}\bar{v}(n+1) = \boldsymbol{PF}\bar{v}(n) + \boldsymbol{P}\bar{q}x(n)$$

Noting that $\hat{\bar{v}}(n+1) = \boldsymbol{P}\bar{v}(n+1)$ and $\bar{v}(n) = \boldsymbol{P}^{-1}\hat{\bar{v}}(n)$, we can write

$$\hat{\bar{v}}(n+1) = (\boldsymbol{PFP}^{-1})\hat{\bar{v}}(n) + (\boldsymbol{P}\bar{q})x(n)$$

Define a new system matrix:

$$\hat{\boldsymbol{F}} = \boldsymbol{PFP}^{-1}$$

By choosing $\boldsymbol{P}$ so that $\hat{\boldsymbol{F}}$ is diagonal, we can simply the system considerably. We can do this using eigenvalues and eigenvectors:

$$\hat{\boldsymbol{F}} = \boldsymbol{PFP}^{-1} = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_N \end{bmatrix}$$

(Another boring review follows - but try doing this on a computer!)

$\boldsymbol{F}\bar{u} = \lambda\bar{u}$ ($\bar{u}$ are eigenvectors)

$$(\boldsymbol{F} - \lambda\boldsymbol{I})\bar{u} = \boldsymbol{0}$$

$$det(\boldsymbol{F} - \lambda\boldsymbol{I}) = 0$$

This yields the characteristic polynomial whose roots, $\{\lambda_i\}$ are the eigenvalues of $\boldsymbol{F}$. For each root λ_i, we have:

$$\boldsymbol{F}\bar{u}_i = \lambda_i\bar{u}_i$$

Define the eigenvector matrix:

$$\boldsymbol{U} = \begin{bmatrix} \bar{u}_1 & \bar{u}_2 & \dots & \bar{u}_N \end{bmatrix}$$

$$\hat{\boldsymbol{F}} = \boldsymbol{U}^{-1}\boldsymbol{FU}$$

This defines a procedure to diagonalize $\boldsymbol{F}$. Importance? How does this relate to random signals and signal models?

(another obscure speech processing reference!)

State-Space Analysis in the Z-Domain

Why? Analogous to what in the analog case?
Recall,

$$\bar{v}(n+1) = \boldsymbol{F}\bar{v}(n) + \bar{q}x(n)$$

In the z-transform domain:

$$z\overline{V}(z) = \boldsymbol{F}\overline{V}(z) + \bar{q}X(z)$$

Solving for $\overline{V}(z)$:

$$\overline{V}(z) = (z\boldsymbol{I} - \boldsymbol{F})^{-1}\bar{q}X(z)$$

Next, we work on the output:

$$y(n) = \bar{g}^t\bar{v}(n) + dx(n)$$

$$Y(z) = [\bar{g}^t(z\boldsymbol{I} - \boldsymbol{F})^{-1}\bar{q} + d]X(z)$$

Hence,

$$H(z) = \frac{Y(z)}{X(z)} = \bar{g}^t(z\boldsymbol{I} - \boldsymbol{F})^{-1}\bar{q} + d$$

$$= \bar{g}^t\sum_{n=1}^{\infty}(\boldsymbol{F}^{n-1}z^{-n})\bar{q} + d$$

Hence, properties of H(z) are closely related to properties of $\boldsymbol{F}$. For example, eigenvalues of $\boldsymbol{F}$ are roots of H(z). What does this say about the relationship of eigenvalues and poles?

State-space descriptions are used primarily in three areas:

- Theoretical studies of DSP systems (research)
- Numerical solutions of Linear systems
- Multiple Input/Multiple Output Systems