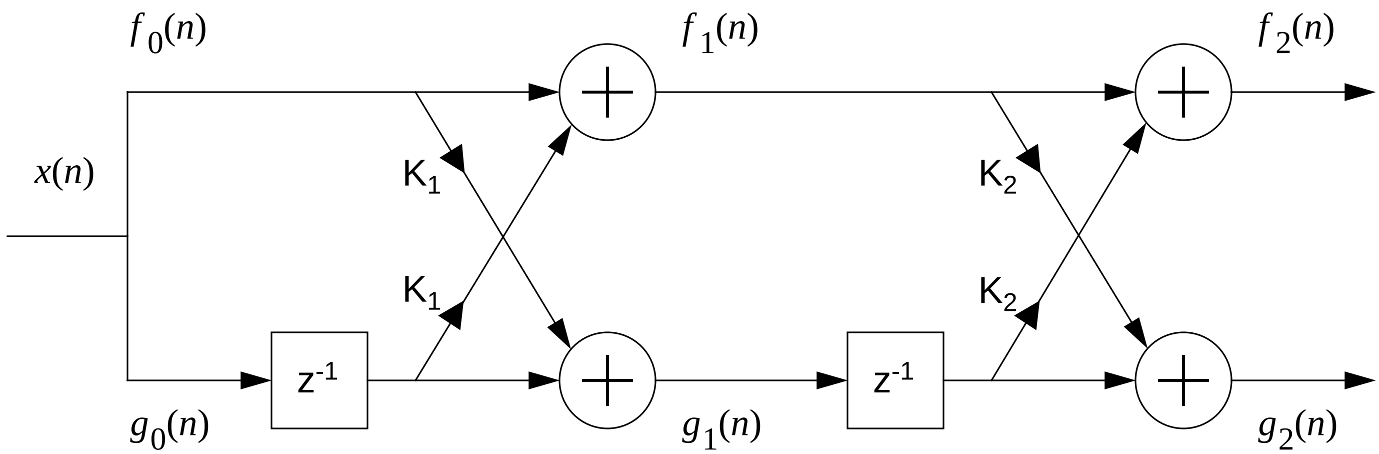
Lattice Filters

Let us define a two-stage lattice filter structure:



We can implement a second-order FIR filter using this structure:

$$y(n) = x(n) + \alpha_2(1)x(n-1) + \alpha_2 x(n-2)$$

The output from the first stage of the lattice is:

$$f_1(n) = x(n) + K_1 x(n-1)$$

$$g_1(n) = K_1 x(n) + x(n-1)$$

The output from the second stage is:

$$f_2(n) = f_1(n) + K_2 g_1(n-1)$$

$$g_2(n) = K_2 f_1(n) + g_1(n-1)$$

We can solve for $f_2(n)$ through substitution of $f_1(n)$ and $g_1(n-1)$:

$$f_2(n) = x(n) + K_1 x(n-1) + K_2[K_1 x(n-1) + x(n-2)]$$

$$= x(n) + K_1(1+K_2)x(n-1) + K_2 x(n-2)$$

If we equate coefficients with our FIR equation above:

$$\alpha_2(2) = K_2 \qquad \alpha_2(1) = K_1(1+K_2)$$

or, solving for $\{K_i\}$,

$$K_2 = \alpha_2(2) \qquad K_1 = \frac{\alpha_2(1)}{1+\alpha_2(2)}$$

General M*th*-Order Lattice Filters

For the general case, we can write:

$$f_0(n) = g_0(n) = x(n)$$

$$f_m(n) = f_{m-1}(n) + K_m g_{m-1}(n-1), \qquad m = 1, 2, \ldots, M-1$$

$$g_m(n) = K_m f_{m-1}(n) + g_{m-1}(n-1), \qquad m = 1, 2, \ldots, M-1$$

The output of the $(M-1)st$ stage corresponds to the output of an FIR filter:

$$y(n) = f_{M-1}(n)$$

This can also be expressed as:

$$f_m(n) = \sum_{k=0}^{m} \alpha_m(k) x(n-k), \qquad \alpha_m(0) = 1$$

We can write this using z-transforms as:

$$F_m(z) = A_m(z) X(z)$$

or,

$$A_m(z) = \frac{F_m(z)}{F_0(z)}$$

Hence, at each stage of the filter, we can view the output as an FIR filter operation. We refer to $A_m(z)$ as the forward filter or forward polynomial. We refer to $\{K_i\}$ as reflection coefficients. $\alpha_i(m)$ are FIR filter coefficients.

Note that for a stable filter:

$$|K_i| < 1$$

Hence, $\{K_i\}$ are much easier to quantize than $\alpha_i(m)$.

We will revisit this formulation in speech processing under the topic of linear prediction.

$G_m(z)$: The Reverse Polynomial

We can similarly solve for $g_2(n)$:

$$\begin{aligned} g_2(n) &= K_2 f_1(n) + g_1(n-1) \\ &= K_2 x(n) + K_1(1+K_2)x(n-1) + x(n-2) \\ &= \alpha_2(2)x(n) + \alpha_2(1)x(n-1) + x(n-2) \end{aligned}$$

Following our previous development, we can see that $g_m(n)$ is the output from an m-stage lattice filter:

$$g_m(n) = \sum_{k=0}^{m} \beta_m(k)x(n-k)$$

where the filter coefficients $\{\beta_m(k)\}$ are associated with a filter whose coefficients are the reverse of $A_m(z)$:

$$\beta_m(k) = \alpha_m(m-k), \qquad k = 0, 1, \ldots, m$$

with $\beta_m(m) = 1$.

In the z-transform domain,

$$G_m(z) = B_m(z)X(z)$$

where

$$\beta_m(z) = \sum_{k=0}^{m} \beta_m(k)z^{-k}$$

and,

$$\beta_m(z) = \sum_{k=0}^{m} \alpha_m(m-k)z^{-k} = z^{-m}A_m(z^{-1})$$

(what does this say about zeros?)

Matrix Formulations: Relationship to Two-Port Networks

We can write the following recurrence relations in the z-domain:

$$F_0(z) = G_0(z) = X(z)$$

$$F_m(z) = F_{m-1}(z) + K_m z^{-1} G_{m-1}(z), \qquad m = 1, 2, \ldots, M-1$$

$$G_m(z) = K_m F_{m-1}(z) + z^{-1} G_{m-1}(z), \qquad m = 1, 2, \ldots, M-1$$

Dividing by $X(z)$:

$$A_0(z) = B_0(z) = 1$$

$$A_m(z) = A_{m-1}(z) + K_m z^{-1} B_{m-1}(z), \qquad m = 1, 2, \ldots, M-1$$

$$B_m(z) = K_m A_{m-1}(z) + z^{-1} B_{m-1}(z), \qquad m = 1, 2, \ldots, M-1$$

This can be written in matrix form as:

$$\begin{bmatrix} A_m(z) \\ B_m(z) \end{bmatrix} = \begin{bmatrix} 1 & K_m \\ K_m & 1 \end{bmatrix} \begin{bmatrix} A_{m-1}(z) \\ z^{-1} B_{m-1}(z) \end{bmatrix}$$

The reflection coefficients can be converted to direct-form filter coefficients by noting that:

$$A_0(z) = B_0(z) = 1$$

$$A_m(z) = A_{m-1}(z) + K_m z^{-1} B_{m-1}(z), \qquad m = 1, 2, \ldots, M-1$$

$$B_m(z) = z^{-m} A_m(z^{-1}), \qquad m = 1, 2, \ldots, M-1$$

Conversions

Conversion from direct-form filter coefficients to lattice coefficients:

$$\alpha_m(0) = 1$$

$$\alpha_m(m) = K_m$$

$$\begin{aligned}\alpha_m(k) &= \alpha_{m-1}(k) + K_m\alpha_{m-1}(m-k) \\ &= \alpha_{m-1}(k) + \alpha_m(m)\alpha_{m-1}(m-k) \qquad 1 \le k \le m-1\end{aligned}$$

$$m = 1, 2, \ldots, M-1$$

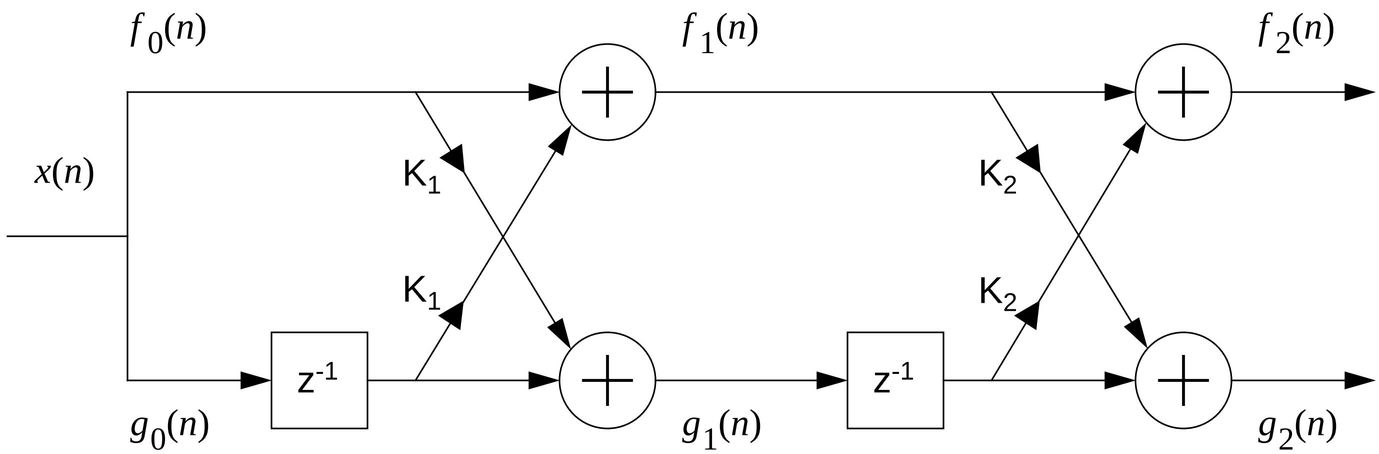
Conversion from lattice coefficients to direct-form filter coefficients:

for m=M-1,M-2,...,1:

$$K_m = \alpha_m(m)$$

$$\begin{aligned}\alpha_{m-1}(k) &= \frac{\alpha_m(k) - K_m\beta_m(k)}{1-K_m^2} \\ &= \frac{\alpha_m(k) - K_m\alpha_m(m-k)}{1-\alpha_m^2(m)} \qquad 1 \le k \le m-1\end{aligned}$$

Forward (Analysis) Filter - All Zeros



Inverse (Synthesis) Filter - All Pole