Computation of the Frequency Response

Recall,

$$H(\omega) = \frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{1 + \sum_{k=1}^{N} a_k e^{-j\omega k}}$$

$H(\omega)$ may be expressed in terms of its poles and zeros by writing it in its factored form:

$$H(\omega) = b_0 \frac{\prod_{k=1}^{M} (1 - z_k e^{-j\omega})}{\prod_{k=1}^{N} (1 - p_k e^{-j\omega})}$$

or, equivalently as

$$H(\omega) = b_0 e^{j\omega(N-M)} \frac{\prod_{k=1}^{M} (e^{j\omega} - z_k)}{\prod_{k=1}^{N} (e^{j\omega} - p_k)}$$

We can express the complex-valued contribution of each pole/zero in polar form as:

$$e^{j\omega} - z_k = V_k(\omega) e^{j\Theta_k(\omega)}$$

$$e^{j\omega} - p_k = U_k(\omega) e^{j\Phi_k(\omega)}$$

where

$$V_k(\omega) \equiv \left| e^{j\omega} - z_k \right| \qquad \Theta_k(\omega) \equiv \angle(e^{j\omega} - z_k)$$

$$U_k(\omega) \equiv \left| e^{j\omega} - p_k \right| \qquad \Phi_k(\omega) \equiv \angle(e^{j\omega} - p_k)$$

Thus, the magnitude of $H(\omega)$ is the product of the magnitudes:

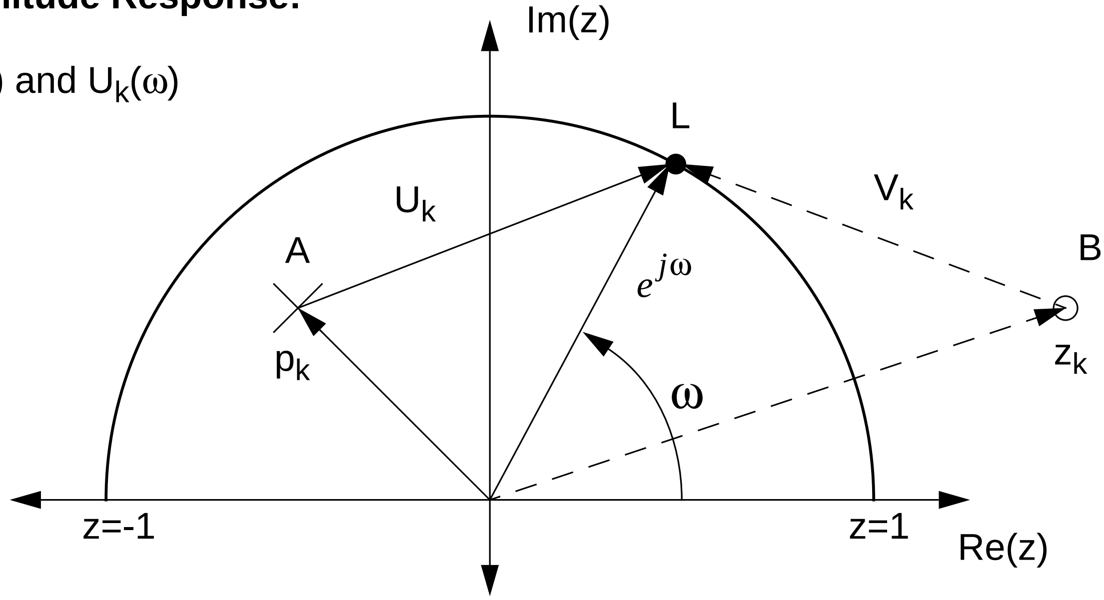
$$H(\omega) = |b_0| \frac{V_1(\omega)V_2(\omega)\ldots V_M(\omega)}{U_1(\omega)U_2(\omega)\ldots U_N(\omega)}$$

The phase of $H(\omega)$ is the sum of the numerator factors minus the sum of the denominator factors:

$$\angle H(\omega) = \angle b_o + \omega(N-M) + \Theta_1(\omega) + \Theta_2(\omega) + \ldots + \Theta_M(\omega)$$
$$-[\Phi_1(\omega) + \Phi_2(\omega) + \ldots + \Phi_N(\omega)]$$

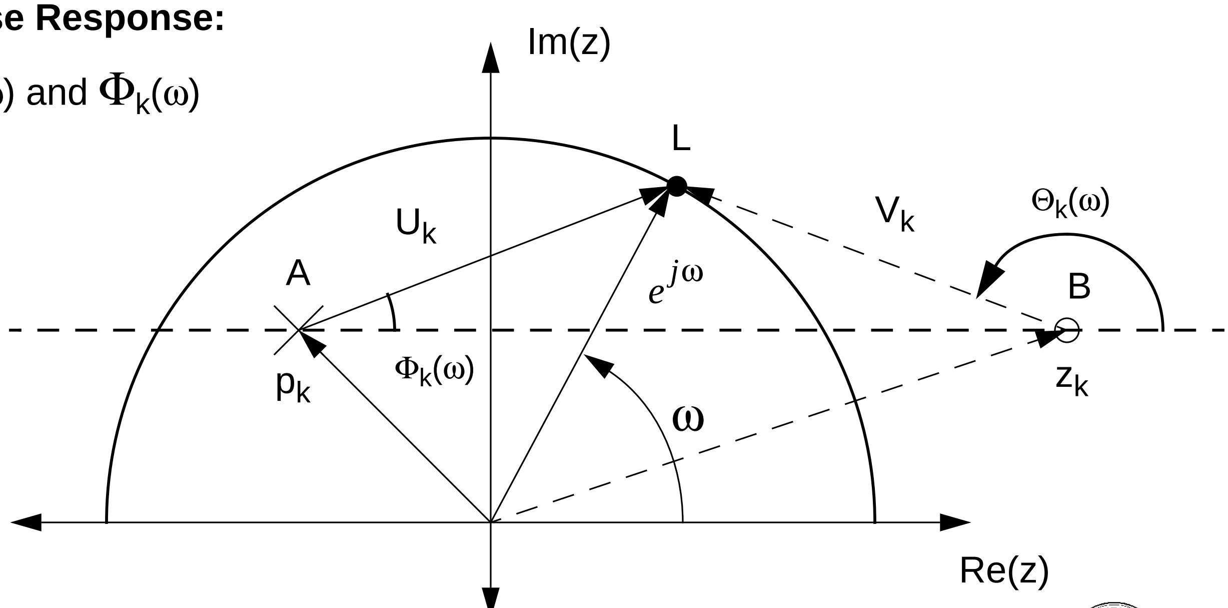
Magnitude Response:

$V_k(\omega)$ and $U_k(\omega)$

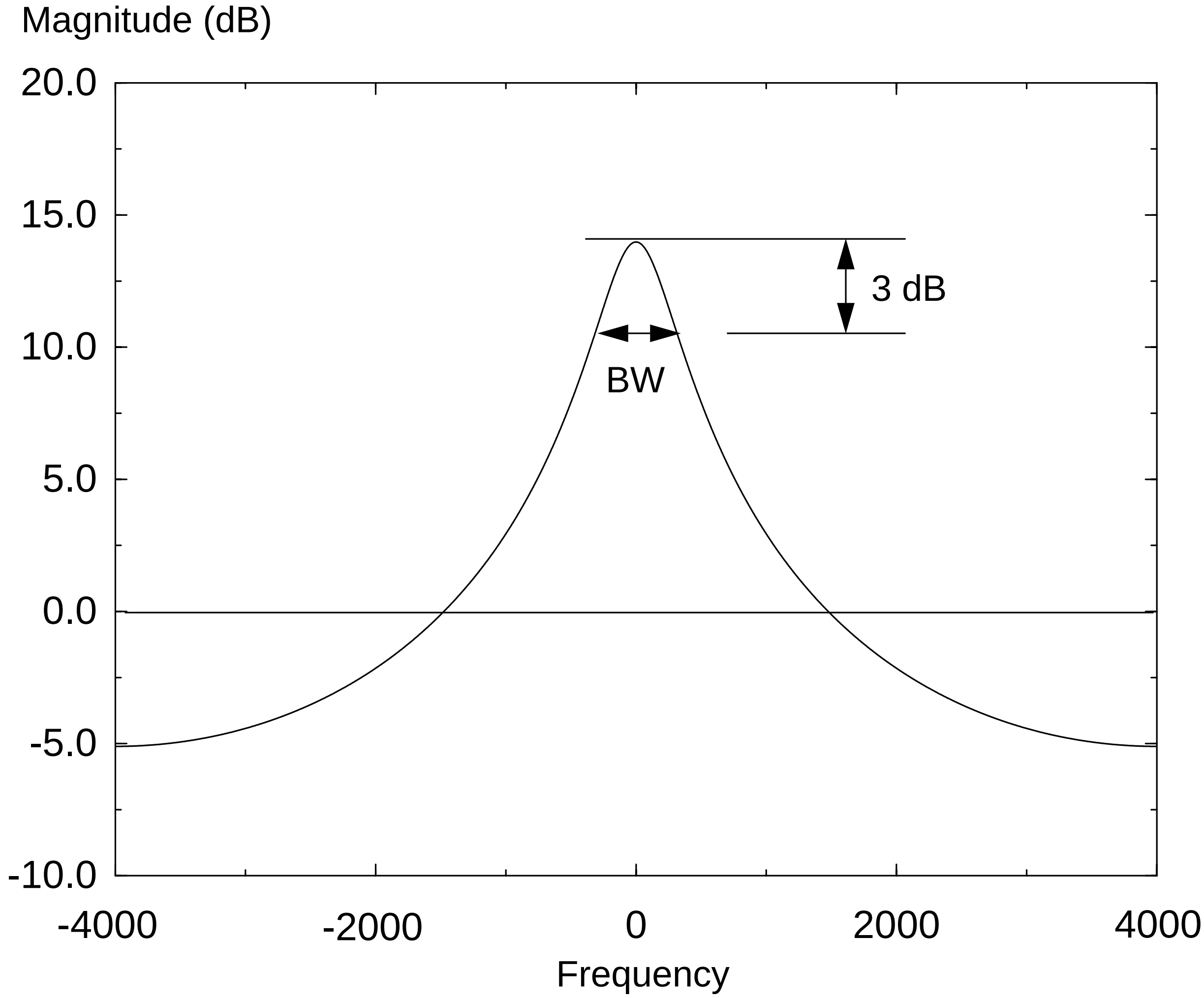


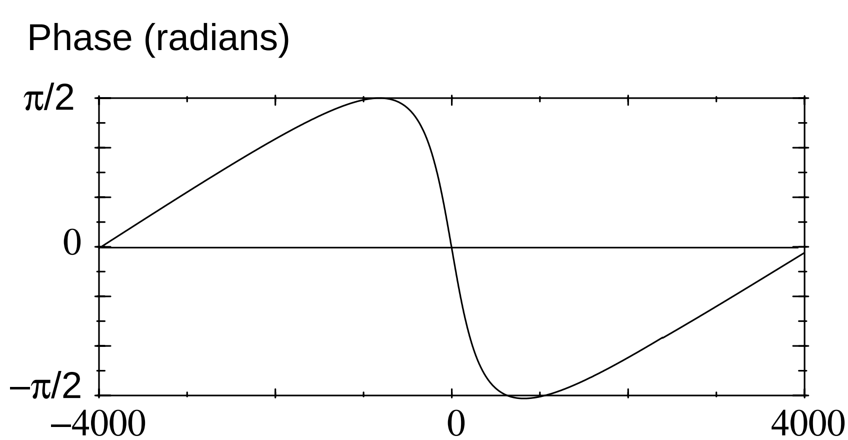
Phase Response:

$\Theta_k(\omega)$ and $\Phi_k(\omega)$



$$h(n) = (0.8)^n u(n) \qquad H(z) = \frac{1}{1 - 0.8z^{-1}} \qquad f_s = 8000 \text{ Hz}$$





First-Order Zeros and Poles

First-order zero:

$$H_z(z) = 1 - az^{-1}$$

$$H_z(\omega) = 1 - re^{j\theta}e^{-j\omega}$$

$$= 1 - r\cos(\omega-\theta) + jr\sin(\omega-\theta)$$

The magnitude response is

$$|H_z(\omega)| = [1 - 2r\cos(\omega-\theta) + r^2]^{1/2}$$

The phase response is:

$$\Theta_z(\omega) = \tan^{-1}\left(\frac{r\sin(\omega-\theta)}{1 - r\cos(\omega-\theta)}\right)$$

The group delay is found by differentiating $\Theta_z(\omega)$:

$$\tau_g(\omega) = \frac{r^2 - r\cos(\omega-\theta)}{1 + r^2 - 2r\cos(\omega-\theta)}$$

These functions are plotted below.

What about:

$$H_z(z) = z^{-1} - az^{-2}$$

What happens for a first-order pole?

Any useful applications of these?

Magnitude and Phase Response For A Zero

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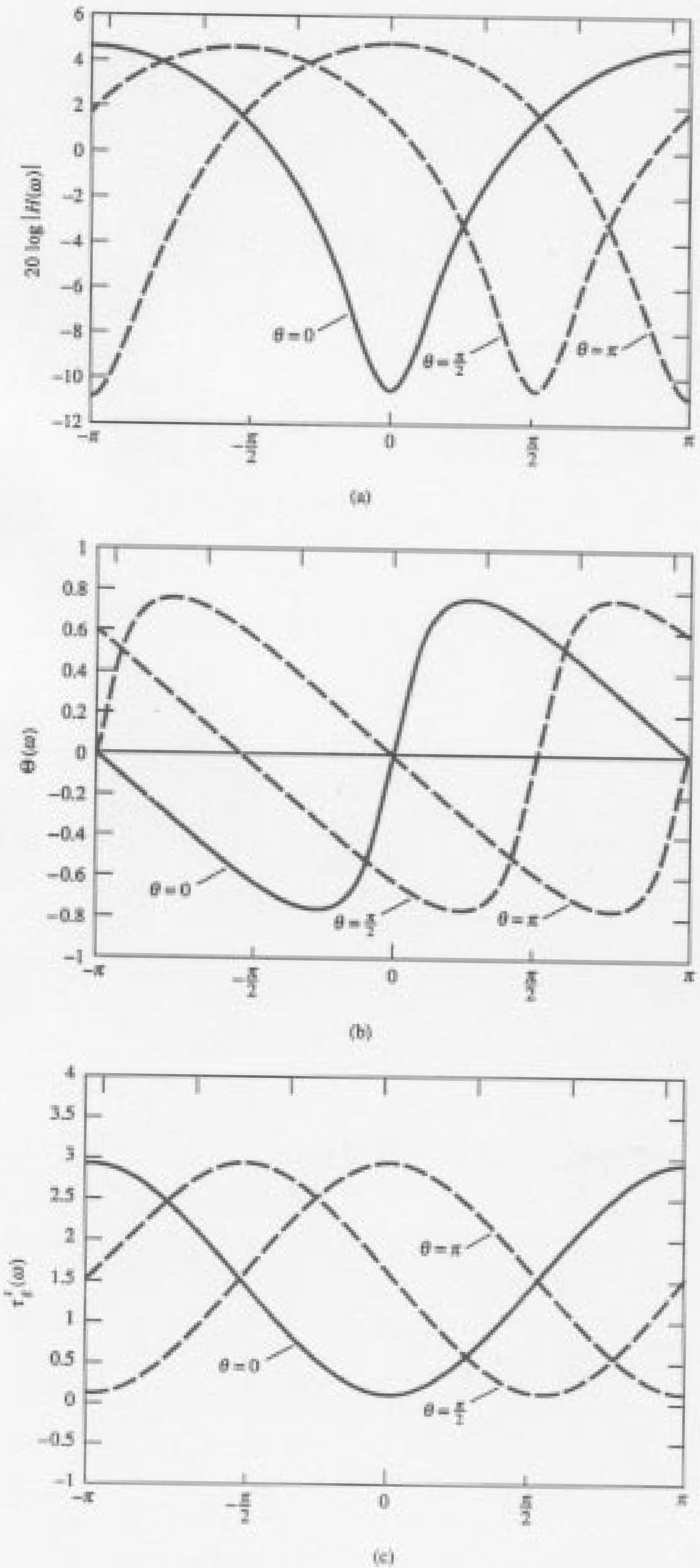


FIGURE 5.9 Magnitude, phase, and group delay for $r = 0.7$, $\theta = 0, \pi/2, \pi$.

Complex-Conjugate Pairs of Poles and Zeros

Consider a system consisting of a complex-conjugate pair of zeros:

$$H_z(z) = (1 - az^{-1})(1 - a^* z^{-1})$$

The magnitude, phase, and group delay functions can be computed as follows:

$$\left|H_z(\omega)\right|_{dB} = 10\log_{10}(1 + r^2 - 2r\cos(\omega - \theta)) + 10\log_{10}(1 + r^2 - 2r\cos(\omega + \theta))$$

$$\Theta_z(\omega) = \tan^{-1}\left(\frac{r\sin(\omega - \theta)}{1 - r\cos(\omega - \theta)}\right) + \tan^{-1}\left(\frac{r\sin(\omega + \theta)}{1 - r\cos(\omega + \theta)}\right)$$

$$\tau_g(\omega) = \frac{r^2 - r\cos(\omega - \theta)}{1 + r^2 - 2r\cos(\omega - \theta)} + \frac{r^2 - r\cos(\omega + \theta)}{1 + r^2 - 2r\cos(\omega + \theta)}$$

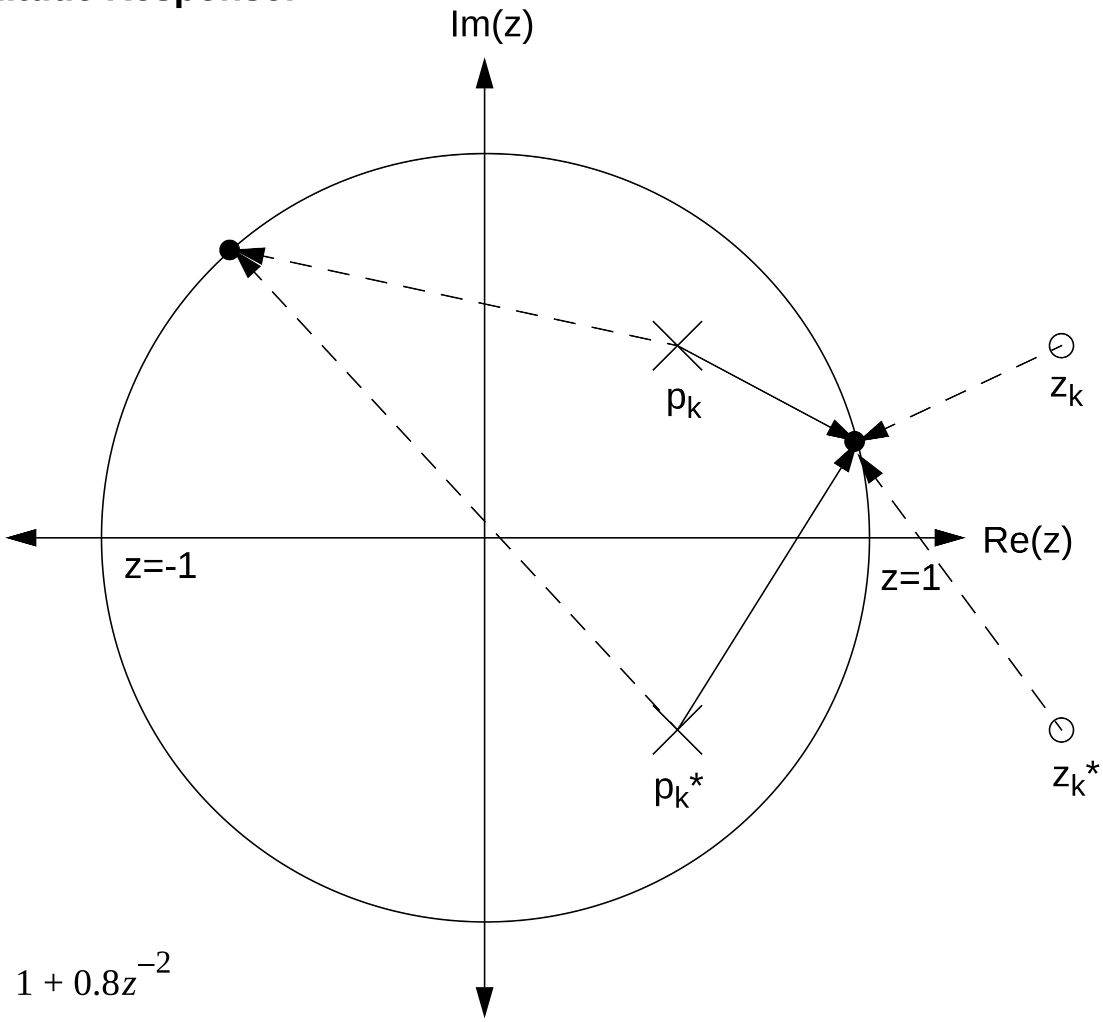
What about complex conjugate poles?

What about a pole-zero combination?

What about a single unstable pole?

A Geometric Interpretation

Magnitude Response:



$$H(z) = 1 + 0.8z^{-2}$$