The System Function and The Frequency Response of LTI Systems (Review)

Recall,

$$H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n}$$

and the output of an LTI system may be expressed in the z-domain as

$$Y(z) = H(z)X(z)$$

For a special class of LTI systems that can be described by a linear, constant-coefficient difference equation of the form

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

the system function is

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

When $\{a_k\} = 0$, $H(z) = \sum_{k=0}^{M} b_k z^{-k}$

and

$$h(n) = \begin{cases} b_n, & 0 \le n \le M \\ 0, & otherwise \end{cases}$$

The system has a finite-duration impulse response (FIR), and its frequency response is composed of all zeros (and poles at the origin).

When $a_k \neq 0$, the system is called an infinite-duration impulse response (IIR) system.

The Frequency Response Function

The Fourier transform relationship between the impulse response and the frequency response function is given by:

$$H(\omega) = \sum_{n=-\infty}^{\infty} h(n)e^{-j\omega n}$$

Recall that this function is periodic with period 2π. The output of an LTI system with frequency response $H(\omega)$ to an aperiodic finite energy signal with Fourier transform $X(\omega)$ is given as:

$$Y(\omega) = H(\omega)X(\omega)$$

The frequency response function is usually expressed in terms of its magnitude $|H(\omega)|$ and its phase $\Theta(\omega)$, where

$$H(\omega) = |H(\omega)|e^{j\Theta(\omega)}$$

usually, the magnitude is plotted on a logarithmic scale as

$$|H(\omega)|_{dB} = 20\log_{10}|H(\omega)| = 10\log_{10}|H(\omega)|^2$$

where the units are decibels (dB). Sometimes we normalize $H(\omega)$ so that its maximum value is unity (zero on the dB scale). Othertimes, we normalize $H(\omega)$ so that its energy is equal to unity.

Example:

$$y(n) = 1.8y(n-1) - 0.81y(n-2) + x(n) + 0.95x(n-1)$$

$$20\log\frac{|H(\omega)|}{|H(\omega)|_{max}} = \frac{1}{195}\frac{\left|1 + 0.95e^{-j\omega}\right|}{\left|1 - 1.8e^{-j\omega} + 0.81e^{-2j\omega}\right|}$$

Relationships Between the System Function and the Frequency Response Function

Recall

$$H(\omega) = H(z)\big|_{z = e^{j\omega}} = \sum_{n=-\infty}^{\infty} h(n)e^{-j\omega n}.$$

In the case where $H(\omega)$ is a rational function of the form $H(z) = B(z)/A(z)$,

$$H(\omega) = b_0 \frac{\prod_{k=1}^{M}(1 - z_k e^{-j\omega})}{\prod_{k=1}^{N}(1 - p_k e^{-j\omega})}$$

It is desirable sometimes to express the magnitude squared as

$$|H(\omega)|^2 = H(\omega)H^*(\omega)$$

For the rational system function,

$$H^*(\omega) = b_0 \frac{\prod_{k=1}^{M}(1 - z_k^* e^{j\omega})}{\prod_{k=1}^{N}(1 - p_k^* e^{j\omega})}$$

It follows that

$$H^*\left(\frac{1}{z^*}\right) = b_0 \frac{\prod_{k=1}^{M}(1 - z_k^* z)}{\prod_{k=1}^{N}(1 - p_k^* z)}$$

Hence, when $h(n)$ is real, the complex-valued poles and zeros occur in complex-conjugate pairs, and $H^*(1/z^*) = H(z^{-1})$, or $H^*(\omega) = H(-\omega)$, so

$$|H(\omega)|^2 = H(\omega)H^*(\omega) = H(\omega)H(-\omega) = H(z)H(\frac{1}{z})\bigg|_{z = e^{j\omega}}$$

If $H(z) = \frac{B(z)}{A(z)}$, and the transforms $D(z) = B(z)B(\frac{1}{z})$, and $C(z) = A(z)A(\frac{1}{z})$ are the z-transforms of the autocorrelation sequences $\{c_l\}$ and $\{d_l\}$, where

$$c_l = \sum_{k=0}^{N-|l|} a_k a_{k+l}, \qquad -N \le l \le N$$

$$d_l = \sum_{k=0}^{M-|l|} b_k b_{k+l}, \qquad -M \le l \le M$$

we can show that $|H(\omega)|^2$ may be expressed as a ratio of polynomial functions of $\cos\omega$:

$$|H(\omega)|^2 = \frac{d_0 + 2\sum_{k=1}^{M} d_k \cos k\omega}{c_0 + 2\sum_{k=1}^{N} c_k \cos k\omega}$$

Note that

$$\cos k\omega = \sum_{m=0}^{k} \beta_m (\cos\omega)^m .$$

Interconnections of LTI Systems

Recall for cascaded systems:

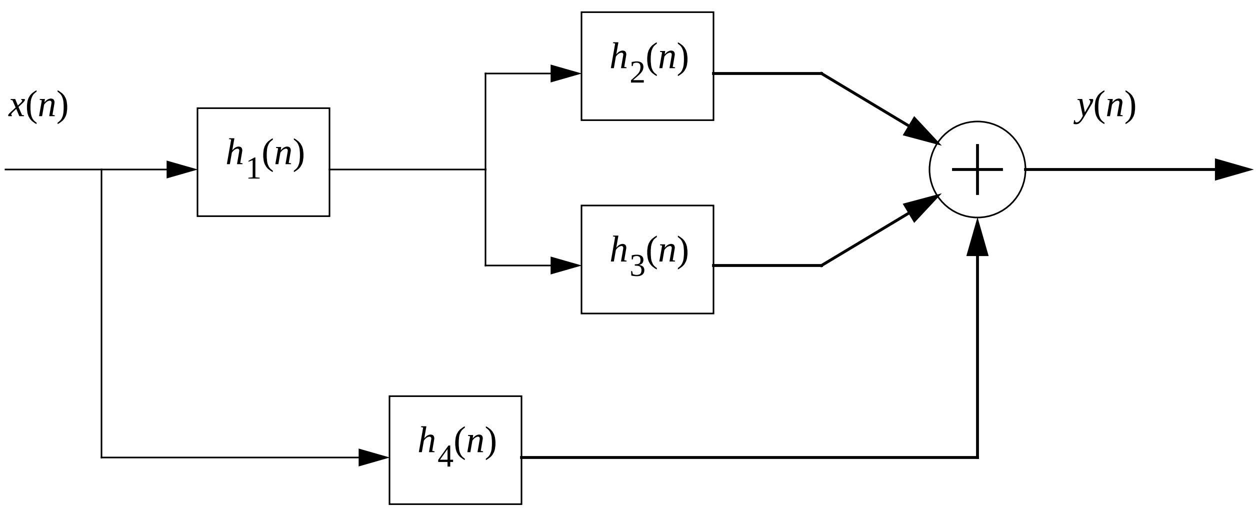
$$H(\omega) = H_1(\omega)H_2(\omega)$$

$$20\log_{10}|H(\omega)| = 20\log_{10}|H_1(\omega)| + 20\log_{10}|H_2(\omega)|$$

$$\Theta(\omega) = \Theta_1(\omega) + \Theta_2(\omega)$$

For parallel systems, no such simple relationships hold.

Example:



System Function:

$$H(z) = H_4(z) + H_1(z)[H_2(z) + H_3(z)]$$

Correlation Functions and Power Spectra

Recall:

$$r_{yy}(m) = r_{hh}(m) \otimes r_{xx}(m)$$
$$r_{yx}(m) = h(m) \otimes r_{xx}(m)$$

Then,

$$S_{yy}(z) = S_{hh}(z)S_{xx}(z)$$
$$= H(z)H(1/z)S_{xx}(z)$$
$$S_{yx}(z) = H(z)S_{xx}(z)$$

and,

$$S_{yy}(\omega) = |H(\omega)|^2 S_{xx}(z)$$

If the input signal is a random signal, the same relationships essentially hold between the signal and the LTI system. With random signals, we often find it convenient to deal with moments, or averages:

$$m_1 = E[x(n)] = \frac{1}{N}\sum_{n=0}^{N} x(n)$$

$$m_2(k) = E[x(n)x(n-k)] = \frac{1}{N}\sum_{n=0}^{N} x(n)x(n-k)$$

$$m_3(k, l) = E[x(n)x(n-k)x(n-l)] = \frac{1}{N}\sum_{n=0}^{N} x(n)x(n-k)x(n-l)$$

$$\ldots$$

What can we say about the moments of a Gaussian process?

Also note that $m_y = m_x H(0) = m_x E[h(n)]$.