

## Relationship of the DFT to Other Transforms

- A periodic sequence  $\{x_p(n)\}$  with fundamental period  $N$  can be represented in a Fourier series of the form:

$$x_p(n) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi kn/N}, \quad -\infty < n < \infty$$

where the Fourier series coefficients are given by the expression

$$c_k = \frac{1}{N} \sum_{n=0}^{N-1} x_p(n) e^{-j2\pi kn/N}, \quad k = 0, 1, 2, \dots, N-1$$

The DFT is therefore related to the Fourier Series by the simple expression:

$$X(k) = Nc_k$$

- For an aperiodic sequence,  $\{x(n)\}$  we have shown that:

$$X(k) = X(\omega)|_{\omega=2\pi k/N} = \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi kn/N}, \quad k = 0, 1, 2, \dots, N-1$$

are the DFT coefficients of the periodic sequence of period  $N$ , given by

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n-lN)$$

and  $x_p(n)$  is determined by aliasing  $x(n)$  over the interval  $0 \leq n \leq N-1$ .

The finite duration sequence

$$\hat{x}(n) = \begin{cases} x_p(n) & 0 \leq n \leq N-1 \\ 0 & \text{otherwise} \end{cases}$$

is obviously different from the original sequence unless  $x(n)$  is finite duration and length  $L \leq N$ , in which case

$$\hat{x}(n) = x_p(n), \quad 0 \leq n \leq N-1$$

Only in this case will the IDFT return  $x(n)$ .

- Recall the z-Transform:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

If we sample this transform at equally spaced points on the unit circle,

$$X(k) \equiv X(z) \Big|_{z=e^{j2\pi k/N}} = \sum_{n=-\infty}^{\infty} x(n)e^{-j2\pi kn/N}, \quad k = 0, 1, 2, \dots, N-1$$

If  $x(n)$  is of finite duration of length  $N$ , its  $z$ -transform is uniquely determined by an  $N$ -point DFT:

$$X(z) = \sum_{n=0}^{N-1} x(n)z^{-n}$$

$$X(z) = \sum_{n=0}^{N-1} \left[ \frac{1}{N} \sum_{k=0}^{N-1} X(k)e^{j2\pi kn/N} \right] z^{-n}$$

which can be simplified to:

$$X(z) = \frac{1-z^{-N}}{N} \sum_{k=0}^{N-1} \frac{X(k)}{1-e^{j2\pi k/N} z^{-1}}$$

This expression is similar to the one we derived for the Fourier transform:

$$X(\omega) = \frac{1-e^{-j\omega N}}{N} \sum_{k=0}^{N-1} \frac{X(k)}{1-e^{-j(\omega-2\pi k)/N}}$$

or,

$$X(\omega) = \sum_{k=0}^{N-1} X\left(\frac{2\pi}{N}k\right)P\left(\omega - \frac{2\pi}{N}k\right), \quad N \geq L$$

- Suppose that  $x_a(t)$  is a continuous-time periodic signal with a fundamental period  $T_p = 1/F_o$ . The signal can be expressed in a Fourier Series

$$x_a(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k F_o t}$$

If we sample  $x_a(t)$  at a uniform rate  $F_s = N/T_p = 1/T$ , we can show that the Fourier series coefficients are related:

$$X(k) = N \sum_{l=-\infty}^{\infty} c_{k-lN} \equiv N\tilde{c}_k$$

These are an aliased version of the Fourier series coefficients  $\{c_k\}$ .

- Recall the relationship of the spectrum of a discrete signal to the Fourier transform of a continuous-time signal:

$$X\left(\frac{F}{F_s}\right) = F_s \sum_{m=-\infty}^{\infty} X_a(F - mF_s), \quad \text{or,} \quad X(\omega) = F_s \sum_{m=-\infty}^{\infty} X_a([\omega - 2\pi m]F_s)$$

We can easily show that:

$$X(k) = F_s \sum_{m=-\infty}^{\infty} X_a\left(\frac{kF_s}{N} - mF_s\right)$$

and,

$$\sum_{l=-\infty}^{\infty} x_a(nT - lNT) \leftrightarrow F_s \sum_{m=-\infty}^{\infty} X_a\left(\frac{kF_s}{N} - mF_s\right)$$

This last relation illustrates time-domain and frequency-domain aliasing.

## Time-Domain Windowing

Let  $\{x(n)\}$  denote a sequence to be analyzed. Let's limit the duration of  $\{x(n)\}$  to  $L$  samples:

$$\hat{x}(n) = x(n)w(n)$$

where  $w(n)$  is a rectangular window and is defined as

$$w(n) = \begin{cases} 1, & 0 \leq n \leq L-1 \\ 0, & \text{otherwise} \end{cases}$$

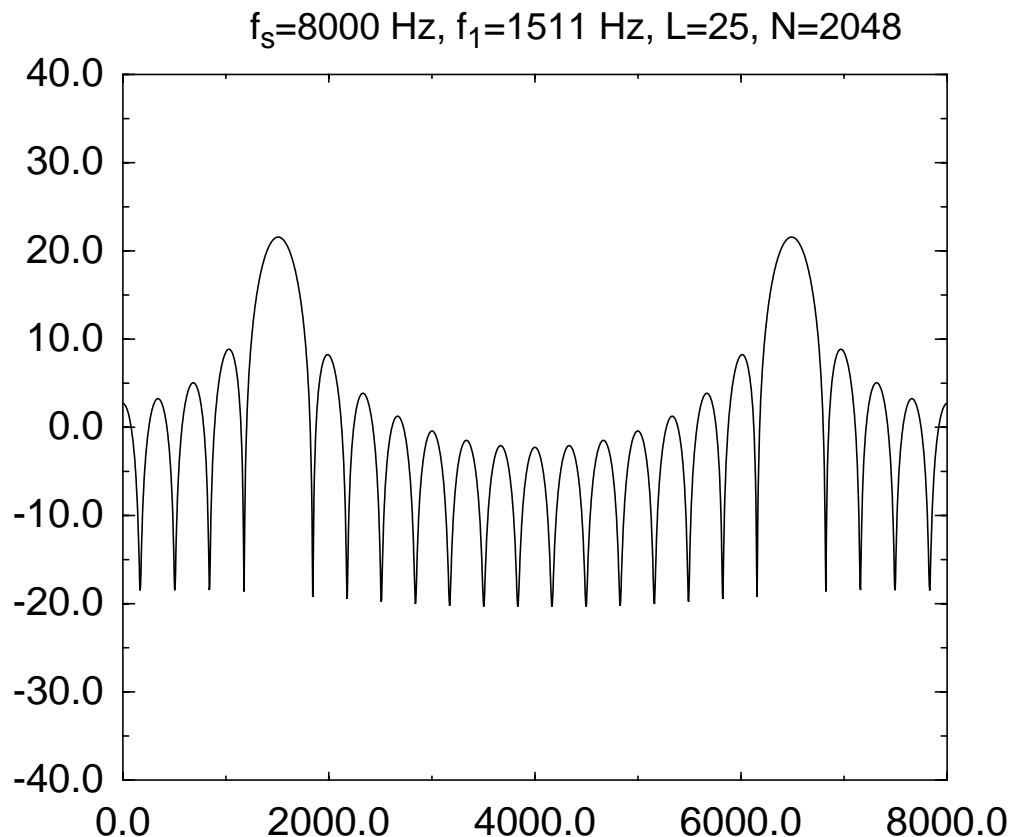
The Fourier transform of  $w(n)$  is given by:

$$W(\omega) = \frac{\sin(\omega(L/2))}{\sin(\omega/2)} e^{-j\omega((L-1)/2)}$$

The transform of  $\hat{x}(n)$  is given by:

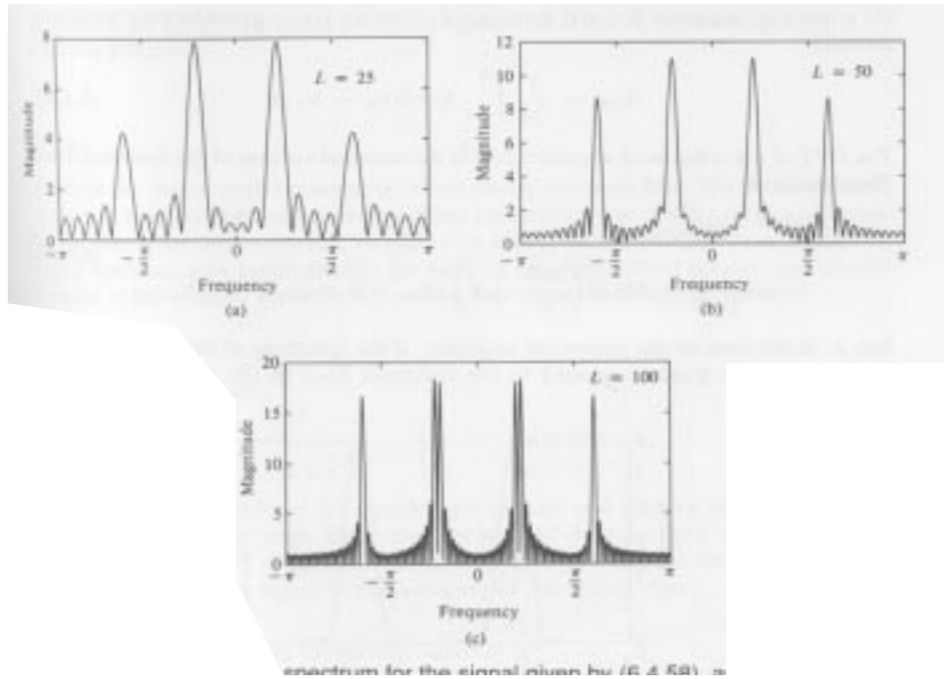
$$\hat{X}(\omega) = \frac{1}{2}[W(\omega - \omega_o) + W(\omega + \omega_o)].$$

This introduces frequency domain aliasing (the so-called picket fence effect):

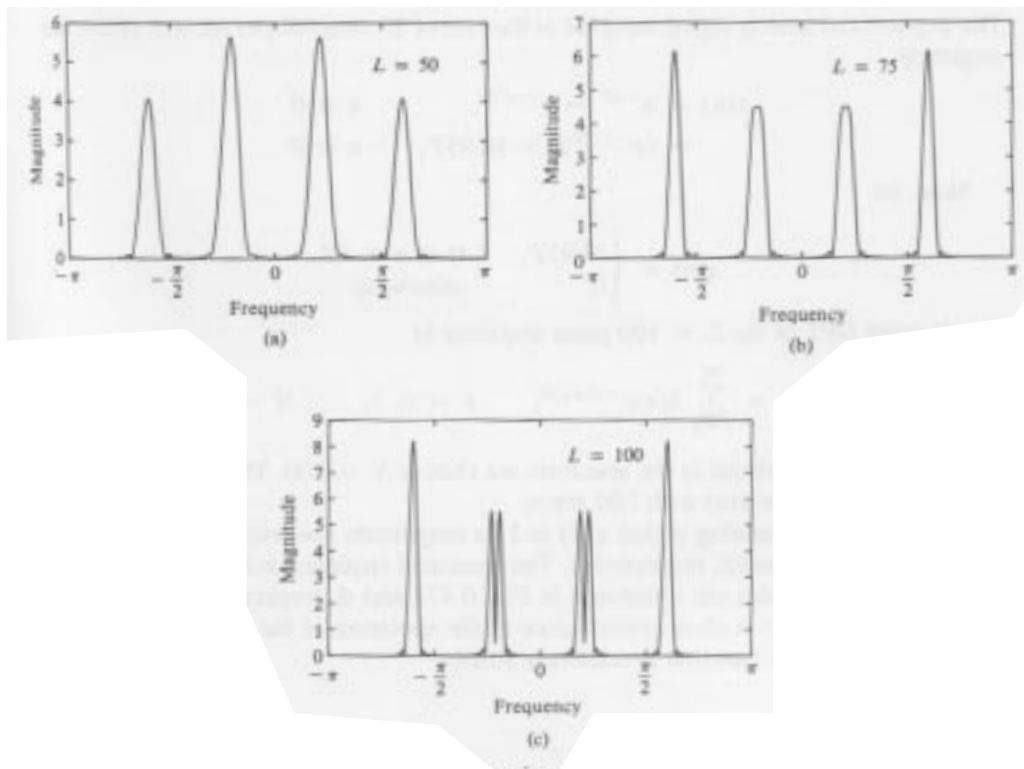


# Improvements Via Better Windows

## Rectangular Window:



## Hanning Window:



## Popular Windows

1. Rectangular: 
$$w(k) = \begin{cases} 1, & |k| \leq N \\ 0, & \textit{otherwise} \end{cases}$$

2. Generalized Hanning: 
$$w_H(k) = w(k) \left[ \alpha + (1 - \alpha) \cos\left(\frac{2\pi}{N}k\right) \right] \quad 0 < \alpha < 1$$

$\alpha = 0.54,$       *Hamming window*  
 $\alpha = 0.50,$       *Hanning window*

3. Bartlett 
$$w_B(k) = w(k) \left[ 1 - \frac{|k|}{N+1} \right]$$

4. Kaiser 
$$w_K(k) = w(k) I_0\left(\alpha \sqrt{1 - \frac{K^2}{N}}\right) / I_0(\alpha)$$

5. Chebyshev: 
$$w_N(k) = 2(x_0^2 - 1)w_{N-1}(k) + x_0^2[w_{N-1}(k-1) + w_{N-1}(k+1)] - w_{N-2}(k)$$

6. Gaussian 
$$w_G(k) = \begin{cases} \exp\left[-\frac{1}{2}k^2 \tan^2\left(\frac{\theta_0}{2}\right)\right] & |k| < N \\ w_G(N-1) / \left[2N \sin^2\left(\frac{\theta_0}{2}\right)\right] & |k| < N \\ 0 & |k| > N \end{cases}$$

There are many others. The most important characteristics are the width of the main lobe and the attenuation in the stop-band (height of highest sidelobe). The Hamming window is used quite extensively.