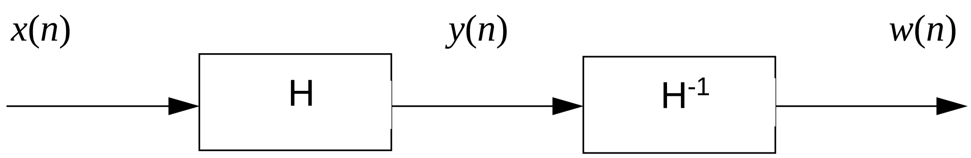
Invertibility of Linear Time-Invariant Systems

- A system is said to be invertible if there is a one-to-one correspondence between its input and output.

Examples?

This implies if we know the output, we can compute the input using linear filtering (obviously useful in recognition applications)



$$w(n) = H^{-1}[y(n)] = H^{-1}[H[x(n)]] = x(n)$$

Note that this is not the case for $y = x^2(n)$.

In the time-domain, this is equivalent to (convolution):

$$w(n) = h_I(n) \otimes h(n) \otimes x(n) = x(n)$$

This implies that $h(n) \otimes h_I(n) = \delta(n)$ and $H_I(z) = \frac{1}{H(z)}$.

$h_I(n)$ is often called the analyzer, or a whitening filter. Why?

If $H(z) = \frac{B(z)}{A(z)}$, then $H_I(z) = \frac{1}{H(z)} = \frac{A(z)}{B(z)}$.

This implies that poles become zeros and zeros become poles.
Hence, we normally like both poles and zeros to be inside the unit circle in practical systems.

This also means if one filter is FIR, the inverse is IIR. Usually, the analyzer is an FIR filter.

Example:

$$h(n) = \left(\frac{1}{2}\right)^n u(n)$$

$$H(z) = \frac{1}{1 - \frac{1}{2}z^{-1}}$$

$$H_I(z) = 1 - \frac{1}{2}z^{-1}$$

$$h_I(n) = \delta(n) - \frac{1}{2}\delta(n-1)$$

Can you prove $h(n) \otimes h_I(n) = \delta(n)$?

Recursion for finding $h_I(n)$ when a closed-form solution for $h(n)$ doesn't exist:

$$\sum_{k=0}^{n} h(k)h_I(n-k) = \delta(n)$$

For causal systems, $h_I(n) = 0$ for $n < 0$. For $n = 0$

$$h_I(0) = \frac{1}{h(0)}$$

For $n > 0$, we have a recursion:

$$h_I(n) = -\sum_{k=1}^{n} \frac{h(k)h_I(n-k)}{h(0)}.$$

This recursion is sensitive to round-off errors because of the accumulation process. As an exercise, compare the poles of a filter found from direct inversion of the transform to the poles of $h_I(n)$ computed from the recursion.

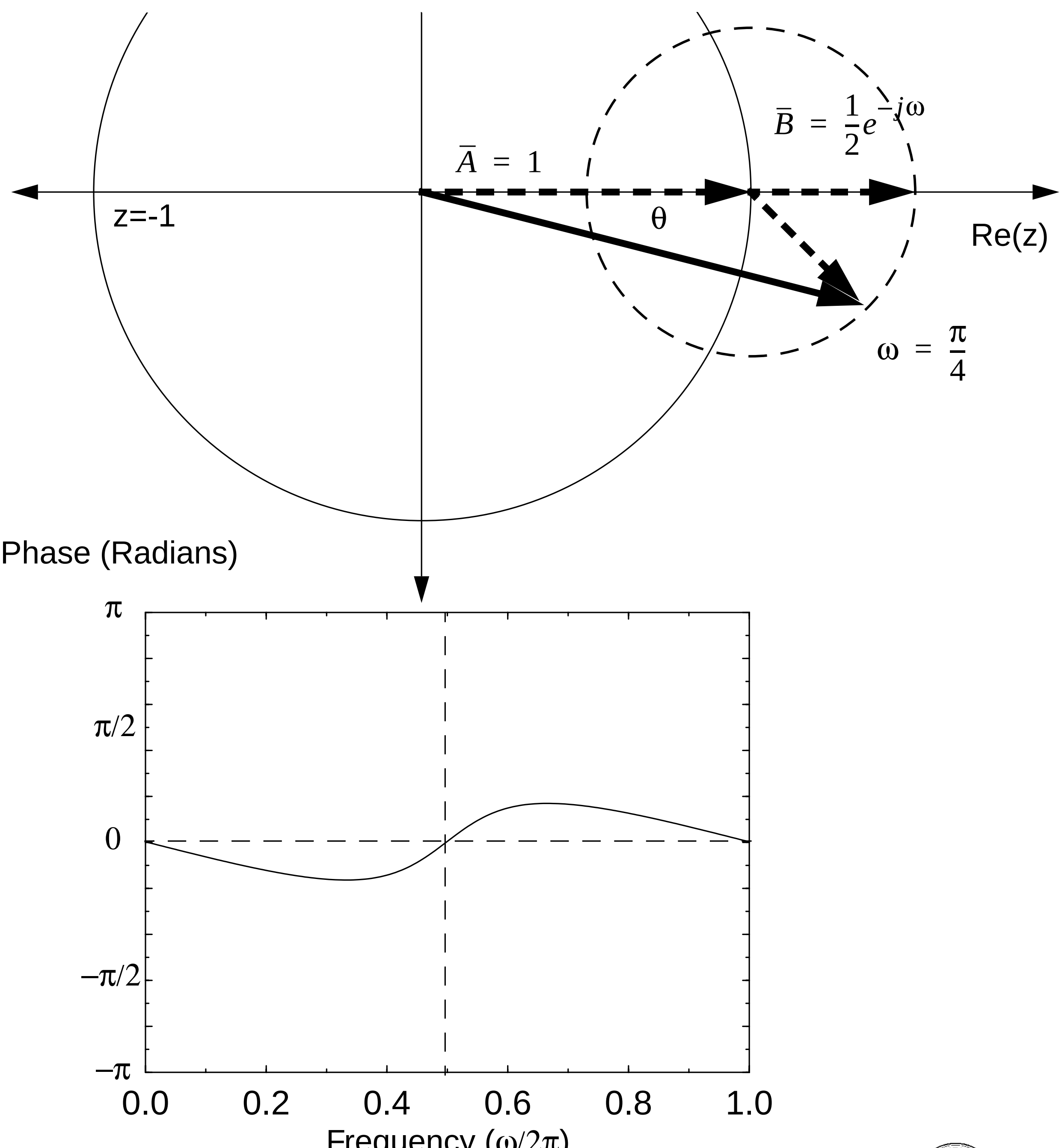
What happens if $h(0) = h(1) = \ldots = h(N) = 0$?

Minimum-Phase, Maximum-Phase, and Mixed-Phase Systems

Consider the following system:

$$H_1(z) = 1 + \frac{1}{2}z^{-1} \quad \left|H_1(\omega)\right| = \sqrt{\frac{5}{4} + \cos\omega} \quad \Theta_1(\omega) = \operatorname{atan}\left(\frac{-\frac{1}{2}\sin\omega}{1 + \frac{1}{2}\cos\omega}\right)$$

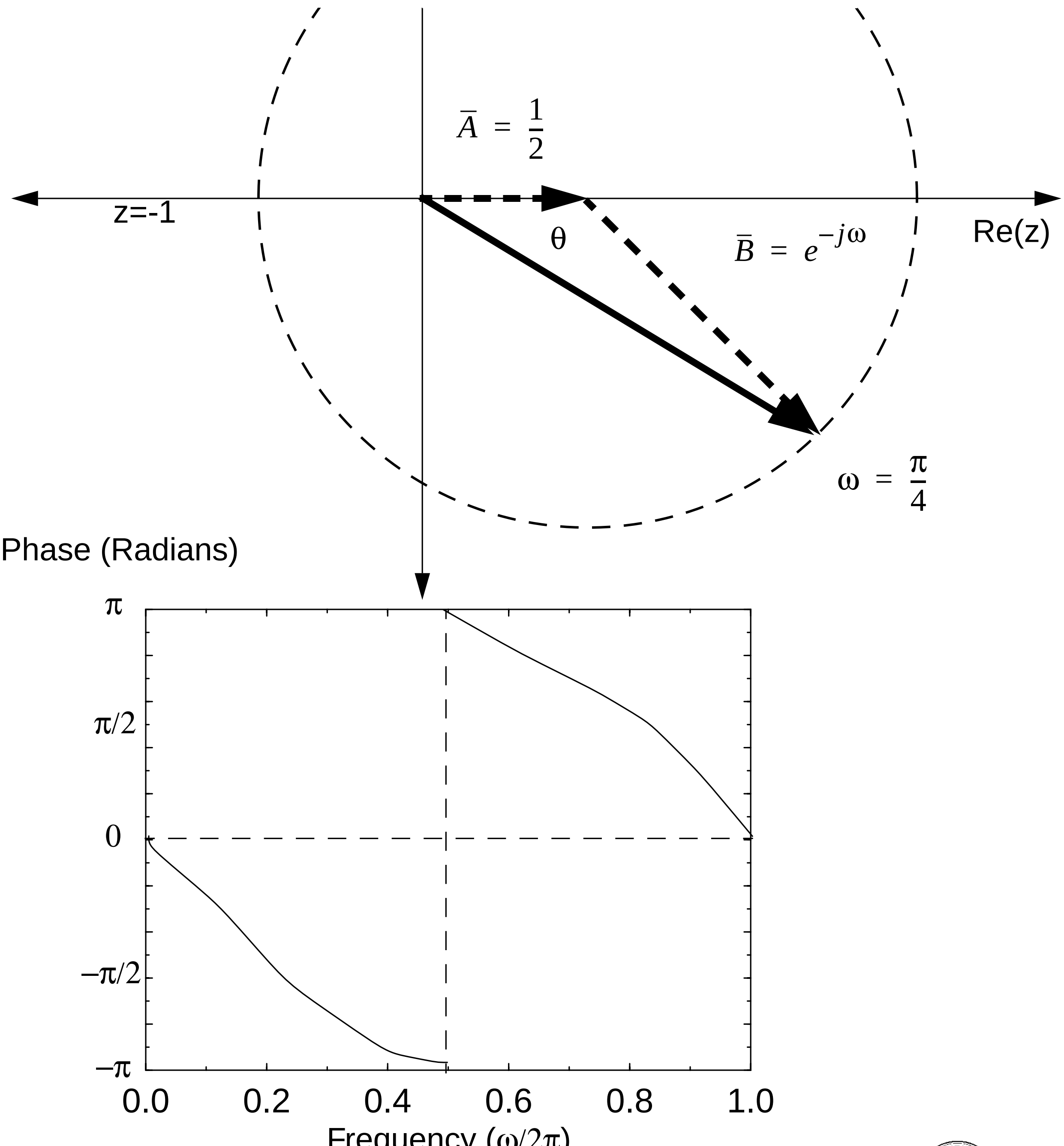
We can depict this graphically:‘



Now, consider the following system:

$$H_2(z) = \frac{1}{2} + z^{-1} \qquad |H_2(\omega)| = \sqrt{\frac{5}{4} + \cos\omega} \qquad \Theta_2(\omega) = \operatorname{atan}\left(\frac{-\sin\omega}{\frac{1}{2} + \cos\omega}\right)$$

We can depict this graphically:‘



Note that these two systems have the same magnitude response but different phase responses.

The net phase change for the first system from $[0, \pi]$ is 0; for the second system it is π. The first system is referred to as *minimum-phase*. The second system is *maximum-phase*.

There are a family of systems that span a continuum between minimum phase and maximum phase that have the same magnitude response.

When computing phase responses, it is important to differentiate between wrapped phase (typically plotted on the interval $[-\pi,\pi]$), and unwrapped phase (which considers phase as a continuous function of frequency and counts the number of complete rotations around the origin).

Further, since phase is proportional to time-delay, there is a difference between wrapped phase and unwrapped phase when converting frequency domain information to the time domain.

An FIR filter composed of all zeros that are inside the unit circle is minimum phase because the net phase change of each zero is zero.

Note also that if we replace a zero, z_k, inside the unit circle, with a zero outside the unit circle, $\frac{1}{z_k}$, the magnitude response doesn't change, but the system becomes non-minimum phase. Hence, there are many realizations of a system with a given magnitude response; one is a minimum phase realization, one is a maximum-phase realization, others are in-between.

Any non-minimum phase pole-zero system can be decomposed into:

$$H(z) = H_{min}(z)H_{ap}(z)$$

We simply need to factor $H(z)$ into its poles and zeros, transform all zeros outside the unit circle, as follows:

$$H(z) = B_1(z)B_2(z) / A(z)$$

$$H_{min}(z) = \frac{B_1(z)B_2(z^{-1})}{A(z)}$$

$$H_{ap}(z) = \frac{B_2(z)}{B_2(z^{-1})}$$

It can be shown that of all the possible realizations of $|H(\omega)|$, the minimum-phase version is the most compact in time:
Define:

$$E(n) = \sum_{k=0}^{n} |h(k)|^2$$

Then, $E_{min}(n) \geq E(n)$ for all n and all possible realizations of $|H(\omega)|$.

System Identification and Deconvolution

Given $x(n)$ and $y(n)$, how do we find the system function? This is one form of the system identification problem.

Method 1:

$$H(z) = \frac{Y(z)}{X(z)} \text{ and } h(n) = Z^{-1}\{H(z)\}$$

Method 2:

$$h(0) = \frac{y(0)}{x(0)}$$

and

$$h(n) = \frac{y(n) - \sum_{k=0}^{n-1} h(k)x(n-k)}{x(n)} \qquad n > 0$$

Method 3:

$$r_{yx}(m) = \sum_{k=0}^{\infty} h(k)r_{xx}(m-k) = h(n) \otimes r_{xx}(m)$$

Compute $r_{yx}(m)$ and solve the deconvolution problem as above:

$$h(n) = \frac{r_{yx}(n) - \sum_{k=0}^{n-1} h(k)r_{xx}(n-k)}{r_{xx}(n)} \qquad n > 0$$

Or, use Fourier transform techniques:

$$H(\omega) = \frac{S_{yx}(\omega)}{S_{xx}(\omega)} = \frac{S_{yx}(\omega)}{|X(\omega)|^2}$$

Homomorphic Deconvolution

$$Y(z) = H(z)X(z)$$

$$\begin{aligned} C_y(z) &= \log Y(z) \\ &= \log H(z) + \log X(z) \end{aligned}$$

In the “time-domain”:

$$c_y(n) = c_h(n) + c_x(n)$$

In many applications, the signals can be separated because the cepstra are non-overlapping.

For example, consider the case where:

$$c_h(n) = 0 \qquad n < M$$

and

$$c_x(n) = 0 \qquad n \geq M.$$

How about a harmonic input, such as glottal air flow during voiced speech, and a system that acts as a lowpass filter with resonances, such as the vocal tract?