Frequency-Domain Characteristics of Linear Time-Invariant Systems

Response to Complex Exponential and Sinusoidal Signals:

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

$$x(n) = Ae^{j\omega n}$$

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)[Ae^{j\omega(n-k)}]$$

$$= A\left[\sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k}\right]e^{j\omega n}$$

Note that the term in brackets is the Fourier Transform of the impulse response:

$$H(\omega) = \sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k}$$

$H(\omega)$ exists if the system is BIBO stable:

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

Hence, the response of the system to a complex exponential is given by:

$$y(n) = AH(\omega)e^{j\omega n}$$

Note that if I vary ω, I can compute the frequency response.
Suppose I apply a “chirp” function?

Example:

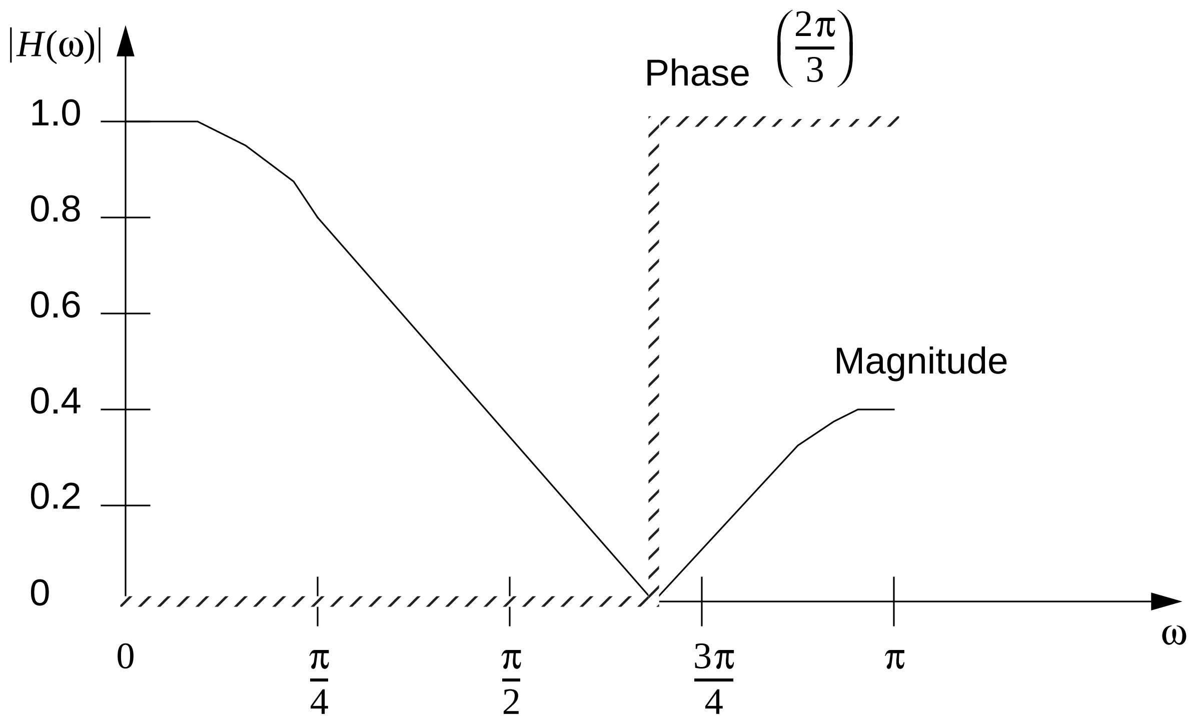
Moving Average Filter (Median Filter):

$$y(n) = \frac{1}{3}[x(n-1) + x(n) + x(n+1)]$$

$$h(n) = \left\{\frac{1}{3}, \mathbf{\frac{1}{3}}, \frac{1}{3}\right\}$$

$$H(\omega) = \frac{1}{3}(e^{j\omega} + 1 + e^{-j\omega}) = \frac{1}{3}(1 + 2\cos\omega)$$

$$|H(\omega)| = \frac{1}{3}|(1 + 2\cos\omega)|$$



Example:

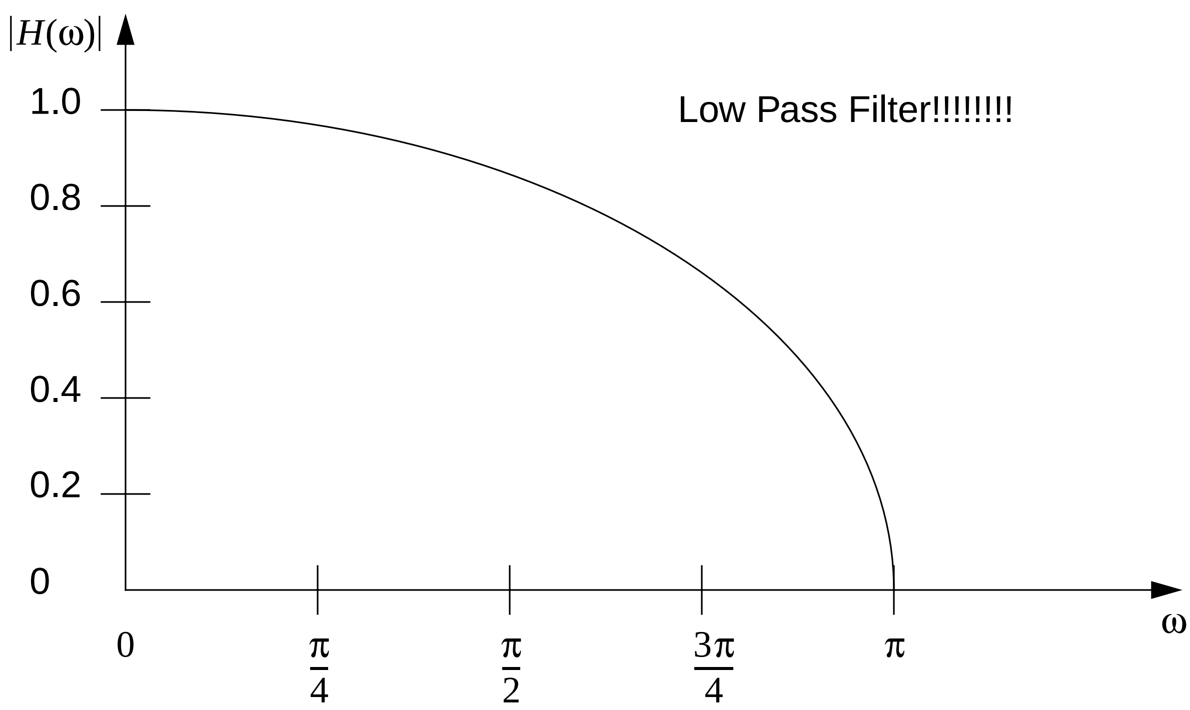
Linear Interpolator:

$$y(n) = \frac{1}{2}[x(n) + x(n-1)]$$

$$h(n) = \left\{\frac{1}{2}, \mathbf{\frac{1}{2}}\right\}$$

$$H(\omega) = \frac{1}{2}(1 + e^{-j\omega})$$

$$\begin{aligned}
|H(\omega)| &= \frac{1}{2}|1 + \cos\omega - j\sin\omega| \\
&= \frac{1}{2}\sqrt{(1+\cos\omega)^2 + (\sin\omega)^2} \\
&= \frac{1}{2}\sqrt{(1 + 2\cos\omega + (\cos\omega)^2) + (\sin\omega)^2} \\
&= \frac{1}{2}\sqrt{2 + 2\cos\omega} \\
&= \frac{1}{\sqrt{2}}\sqrt{1+\cos\omega}
\end{aligned}$$



Example:

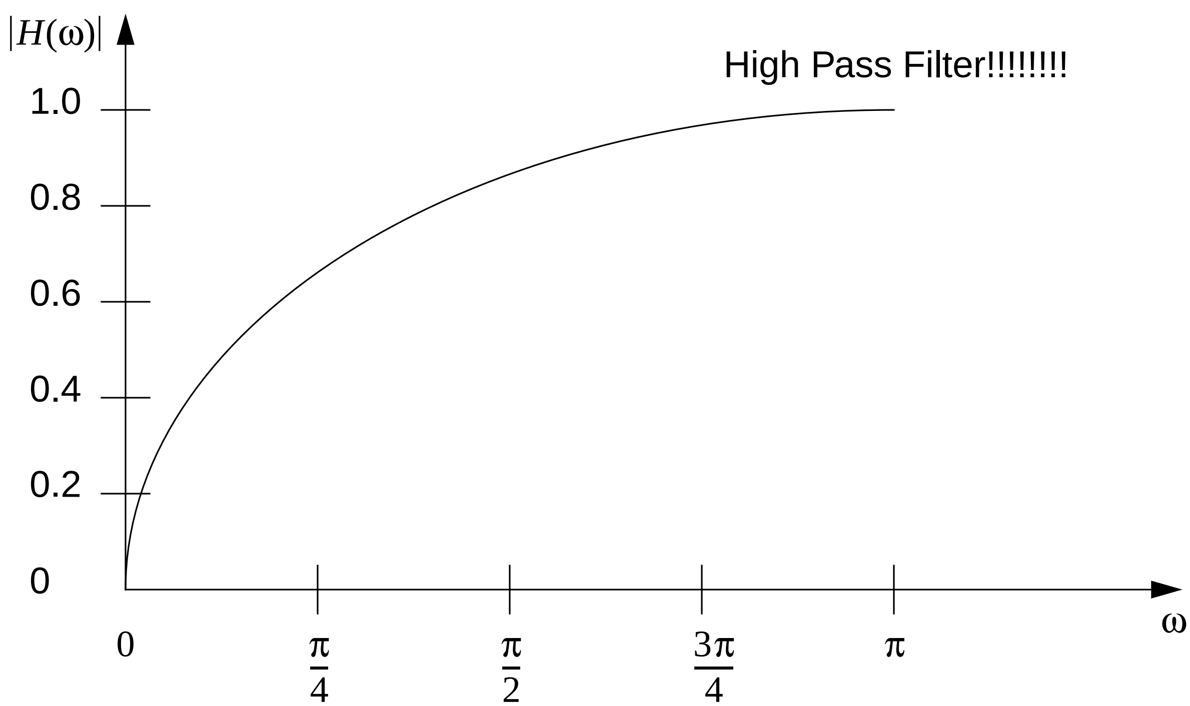
Differentiator:

$$y(n) = \frac{1}{2}[x(n) - x(n-1)]$$

$$h(n) = \left\{-\frac{1}{2}, \mathbf{\frac{1}{2}}\right\}$$

$$H(\omega) = \frac{1}{2}(1 - e^{-j\omega})$$

$$\begin{aligned} |H(\omega)| &= \frac{1}{2}|1 - \cos\omega + j\sin\omega| \\ &= \frac{1}{2}\sqrt{(1-\cos\omega)^2 + (\sin\omega)^2} \\ &= \frac{1}{2}\sqrt{(1 - 2\cos\omega + (\cos\omega)^2) + (\sin\omega)^2} \\ &= \frac{1}{2}\sqrt{2 - 2\cos\omega} \\ &= \frac{1}{\sqrt{2}}\sqrt{1 - \cos\omega} \end{aligned}$$



Steady-State and Transient Response to Sinusoidal Input Signals

Consider:

$$y(n) = ay(n-1) + x(n)$$

Its response is given by:

$$y(n) = a^{n+1}y(-1) + \sum_{k=0}^{n} a^k x(n-k) \qquad n \geq 0$$

Suppose:

$$x(n) = Ae^{j\omega n} \qquad n \geq 0$$

It can be shown:

$$y(n) = a^{n+1}y(-1) - \frac{Aa^{n+1}e^{-j\omega(n+1)}}{1-ae^{-j\omega}}e^{j\omega n} + \frac{A}{1-ae^{-j\omega}}e^{j\omega n} \qquad n \geq 0$$

Note that:

$$y_{ss}(n) = \lim_{n \to \infty} y(n) = \frac{A}{1-ae^{-j\omega}}e^{j\omega n}$$

$$= AH(\omega)e^{j\omega n}$$

and,

$$y_{tr}(n) = a^{n+1}y(-1) - \frac{Aa^{n+1}e^{-j\omega(n+1)}}{1-ae^{-j\omega}}e^{j\omega n} \qquad n \geq 0$$

Steady-State Response to Periodic Input Signals:

$$x(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N}$$

$$y_k(n) = c_k H(\frac{2\pi}{N}k) e^{j2\pi kn/N}$$

$$y(n) = \sum_{k=0}^{N-1} c_k H(\frac{2\pi}{N}k) e^{j2\pi kn/N}$$

Response is periodic with same period as input. The amplitude and phases of the output sinusoidal components can be changed.

Response to Aperiodic Input Signals: