The Fourier Series of a Continuous-Time Periodic Signal

Define:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k F_0 t}$$

where $F_0 = \frac{1}{T_p}$ is the fundamental frequency (period).

The coefficients, c_k, can be shown to be given by the following expression:

$$c_k = \frac{1}{T_p} \int_{T_p} x(t) e^{-j2\pi k F_0 t} dt$$

The signal $x(t)$ can be EXACTLY recovered from these coefficients using the above definition.

The Dirichlet conditions guarantee convergence at every value of $x(t)$ except at values of t for which $x(t)$ is discontinuous (in which case it converges to the average value). The Dirichlet conditions are:

1. The signal has a finite number of finite discontinuities in any period.

2. The signal has a finite number of maxima and minima during any period..

3. The signal is absolutely integrable in any period:

$$\int_{-\infty}^{\infty} |x(t)| dt < \infty .$$

What is the utility of this representation?

In general, Fourier coefficients are complex:

$$c_k = |c_k| e^{j\theta_k}$$

For periodic signals that are real, c_k and c_{-k} are complex conjugates. In this case, we may write the Fourier series as:

$$x(t) = c_0 + 2 \sum_{k=1}^{\infty} |c_k| \cos(2\pi k F_0 t + \theta_k)$$

what is the meaning of c_0?

We may also write the above expression as:

$$x(t) = a_0 + \sum_{k=1}^{\infty} (a_k \cos(2\pi k F_0 t) - b_k \sin(2\pi k F_0 t))$$

where

$$a_0 = c_0$$
$$a_k = 2|c_k| \cos\theta_k.$$
$$b_k = 2|c_k| \sin\theta_k$$

The Power Density Spectrum of Periodic Signals:

A periodic signal has infinite energy and a finite average power defined as:

$$P_x = \frac{1}{T_p} \int_{T_p} |x(t)|^2 dt$$

$$P_x = \frac{1}{T_p} \int_{T_p} x(t) \left(\sum_{k=-\infty}^{\infty} c_k^* e^{-j2\pi k F_0 t} \right) dt$$

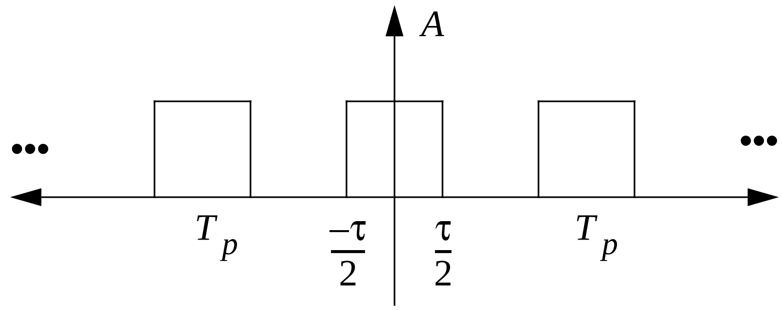
This can be shown to simplify to:

$$P_x = \sum_{k=-\infty}^{\infty} |c_k|^2 .$$

If the periodic signal is real:

$$\begin{aligned} P_x &= c_0^2 + 2 \sum_{k=1}^{\infty} |c_k|^2 \\ &= c_0^2 + \frac{1}{2} \sum_{k=1}^{\infty} (a_k^2 + b_k^2) \end{aligned}$$

Example:



$$c_0 = \frac{1}{T_p} \int_{-T_p/2}^{T_p/2} x(t)dt = \frac{A\tau}{T_p}$$

Define $F_0 = \frac{1}{T_p}$:

$$c_k = \frac{1}{T_p} \int_{-T_p/2}^{T_p/2} Ae^{-j2\pi kF_0t} dt$$

$$= \frac{A\tau}{T_p} \frac{\sin \pi kF_0\tau}{\pi kF_0\tau}, \qquad k = \pm 1, \pm 2, \ldots$$

Note that this is a LINE SPECTRUM with a sinc function envelope. Zero crossings occur at $k = l\frac{T_p}{\tau}$.

What happens if I increase τ? increase T_p?

Frequency Analysis of Continuous-Time Aperiodic Signals: The Fourier Transform

Analysis:

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt$$

Synthesis (Inversion):

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df$$

A Fourier transform is guaranteed to exist if the *Dirichlet conditions* hold:

1. The signal has a finite number of finite discontinuities.
2. The signal has a finite number of maxima and minima.
3. The signal is absolutely integrable:

$$\int_{-\infty}^{\infty} |x(t)| dt < \infty .$$

Energy Density Spectrum of Aperiodic SIgnals

Let $x(t)$ be any finite energy signal with a Fourier transform $X(f)$.

Its energy is:

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

Note that:

$$\begin{aligned}
E_x &= \int_{-\infty}^{\infty} |x(t)|^2 dt \\
&= \int_{-\infty}^{\infty} x(t) x^*(t) dt \\
&= \int_{-\infty}^{\infty} x(t) \left(\int_{-\infty}^{\infty} X^*(f) e^{-j2\pi ft} df \right) dt \\
&= \int_{-\infty}^{\infty} X^*(f) \left[\int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt \right] df \\
&= \int_{-\infty}^{\infty} X^*(f) X(f) df \\
&= \int_{-\infty}^{\infty} |X(f)|^2 df
\end{aligned}$$

Therefore, we have *Parseval's relation* for aperiodic, finite energy signals:

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df$$

We define the energy density spectrum of $x(t)$ as:

$$S_{xx} = |X(f)|^2$$

Note: a real function, no phase information.

Example:

$$x(t) = \begin{cases} A, & |t| \le \tau/2 \\ 0, & |t| > \tau/2 \end{cases}$$

$$X(f) = A\tau\frac{\sin\pi f\tau}{\pi f\tau}$$

Note: zero crossings every $1/\tau$.

Question(s): (a) Suppose I shift $x(t)$ forward by $\tau/2$, what happens to its energy density?

(b) Suppose I increase τ, what happens to $X(f)$?