Response of Systems with Rational System Functions

Consider a linear constant-coefficient difference equation that can be modeled as poles and zeros (a rational system function). Let us assume that the input signal can also be decomposed as poles and zeros:

$$X(z) = \frac{N(z)}{Q(z)} \qquad H(z) = \frac{B(z)}{A(z)} \qquad Y(z) = H(z)X(z) = \frac{B(z)N(z)}{A(z)Q(z)}$$

Using a partial fractions expansion, $Y(z)$ can be factored into two terms:

$$Y(z) = \sum_{k=1}^{N} \frac{A_k}{1 - p_k z^{-1}} + \sum_{k=1}^{L} \frac{Q_k}{1 - q_k z^{-1}}$$

The inverse z-transform gives:

$$y(n) = \sum_{k=1}^{N} A_k (p_k)^n u(n) + \sum_{k=1}^{L} Q_k (q_k)^n u(n)$$

- The first part is the natural response of the system and is a function of the poles of the system.
- The second part is the forced response of the system and is a function of the poles of the input signal.

Note that if the system is stable $(|p_k| < 1)$, the first term decays to zero. This is also called the transient response.

The second term, denoted the forced response, will decay to zero if the poles are inside the unit circle. If the poles are on the unit circle (a marginally stable signal), we will obtain the steady state response (the response to a sinewave).

Response of Systems with Non-Zero Initial Conditions

Suppose that $x(n)$ is causal and is applied to a system with non-zero initial conditions at $n = 0$:

$$Y^{+}(z) = -\sum_{k=1}^{N} a_k z^{-k}\left[Y^{+}(z) + \sum_{n=1}^{k} y(-n) z^{-n}\right] + \sum_{k=0}^{M} b_k z^{-k} X^{+}(z)$$

Since $x(n)$, $X^{+}(z) = X(z)$, and we can re-write $Y^{+}(z)$ as:

$$Y^{+}(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} X(z) - \frac{\sum_{k=1}^{N} a_k z^{-k} \sum_{n=1}^{k} y(-n) z^{n}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

$$= H(z)X(z) + \frac{N_0(z)}{A(z)}$$

where

$$N_0(z) = -\sum_{k=1}^{N} a_k z^{-k} \sum_{n=1}^{k} y(-n) z^{n}$$

The output of the system can be subdivided into two parts, the zero-state response ($H(z)X(z)$ and the component due to the non-zero initial conditions $\left(\frac{N_0(z)}{A(z)}\right)$. The total response is the sum of these two.

Causality and Stability

A causal system satisfies the condition $h(n) \neq 0, \qquad n > 0$.
The ROC of the z-transform is the exterior of a circle. We formally state this:

> A linear time-invariant system is causal if and only if the ROC of the system function is the exterior of a circle of radius $r < \infty$, including the point $z = \infty$.

A necessary and sufficient condition for BIBO stability is $\sum_{n=-\infty}^{\infty} |h(n)| < \infty$.

This implies that $H(z)$ must contain the unit circle within its ROC:

$$|H(z)| = \left| \sum_{n=-\infty}^{\infty} h(n) z^{-n} \right|$$

it follows that

$$|H(z)| \leq \sum_{n=-\infty}^{\infty} \left| h(n) z^{-n} \right| = \sum_{n=-\infty}^{\infty} |h(n)| \left| z^{-n} \right|$$

when evaluated on the unit circle $(|z| = 1)$:

$$|H(z)| \leq \sum_{n=-\infty}^{\infty} |h(n)|$$

- A linear time-invariant system is BIBO stable if and only if the ROC of the system function includes the unit circle.

Since the ROC cannot contain any poles of $H(z)$,

- A causal linear time-invariant system is BIBO stable if and only if all the poles of $H(z)$ are inside the unit circle.

Multiple-Order Poles and Stability (Bounded Input and A Marginally Stable System Does Not Mean Bounded Output)

Consider:

$$y(n) = y(n-1) + x(n) \qquad x(n) = u(n)$$

$$H(z) = \frac{1}{1-z^{-1}} \qquad X(z) = \frac{1}{1-z^{-1}}$$

$$Y(z) = \frac{1}{(1-z^{-1})^2}$$

$$y(n) = (n+1)u(n)$$

Note that even though $u(n)$ is bounded, the output is a ramp and unbounded. This is because the poles of both $u(n)$ and $h(n)$ are on the unit circle, which makes both marginally stable (recalling a term from Laplace transform theory).

The SchÜr-Cohn Stability Test

Consider:

$$A(z) = 1 + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_N z^{-N}$$

Define:

$$A_m(z) = \sum_{k=0}^{m} a_m(k) z^{-k}, \qquad a_m(0) = 1$$

Define a reverse polynomial:

$$\begin{aligned} B_m(z) &= z^{-m} A_m(z^{-1}) \\ &= \sum_{k=0}^{m} a_m(m-k) z^{-k} \end{aligned}$$

The SchÜr-Cohn Stability test states that the polynomial $A_m(z)$ has all of its roots inside the unit circle if and only if the reflection coefficients, K_m, satisfy the condition $|K_m| < 1$ for all $m = 1, 2, \ldots, N$. The reflection coefficients are found from the following recursion:

$$A_N(z) = A(z)$$

$$K_N = a_N(N)$$

$$A_{m-1}(z) = \frac{A_m(z) - K_m B_m(z)}{1 - K_m^2}$$

where $K_m = a_m(m)$.

The details will be left to Example 4.6.7 (and a computer assignment).

Converting From Reflection Coefficients to Predictor Coefficients

Not surprisingly, we can convert from reflection to predictor coefficients:

$$a_N(k) = a_k, \qquad k = 1, 2, \ldots, N$$

for $m = N, N-1, \ldots, 1$, compute

$$K_m = a_m(m), \qquad a_{m-1}(0) = 1$$

and

$$a_{m-1}(k) = \frac{a_m(k) - K_m b_m(k)}{1 - K_m^2}, \qquad k = 1, 2, \ldots, m-1$$

where

$$b_m(k) = a_m(m-k), \qquad k = 0, 1, \ldots, m$$

See the paper on Signal Modeling that I passed out in class.

Stability of Second Order Systems

Consider the system:

$$y(n) = -a_1 y(n-1) - a_2 y(n-2) + b_0 x(n)$$

$$H(z) = \frac{b_0}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

$$p_{1,2} = -\frac{a_1}{2} \pm \sqrt{\frac{a_1^2 - 4a_2}{4}}$$

Note:

$$a_1 = -(p_1 + p_2)$$

$$a_2 = p_1 p_2$$

Stability conditions:

$$|a_2| < 1$$

$$|a_1| < 1 + a_2$$

Real and Distinct Poles $(a_1^2 > 4a_2)$:

$$h(n) = \frac{b_0}{p_1 - p_2}(p_1^{n+1} - p_2^{n+1})u(n)$$

(double decaying exponential)

Real and Equal Poles $a_1^2 = 4a_2$:

$$h(n) = b_0 (n+1) p^n u(n)$$

(product of a ramp and a decaying exponential)

Complex Conjugate Poles $(a_1^2 < 4a_2)$:

(a damped sinusoid)

$$h(n) = \frac{b_0}{\sin \omega_0} \sin((n+1)\omega_0) u(n)$$