The One-Sided Z-Transform

The one-sided or unilateral z-transform is defined by

$$X^{+}(z) \equiv \sum_{n=0}^{\infty} x(n) z^{-n}$$

Other notations used are $Z^{+}\{x(n)\}$ and

$$x(n) \overset{z^{+}}{\longleftrightarrow} X^{+}(z)$$

The one-sided differs from the two-sided z-transform only in the lower limit. This results in the following characteristics that are different from the two-sided transform:

(1) It does not contain information about the signal for negative values of time ($n < 0$).

(2) It is unique only for causal signals.

(3) The one-sided z-transform is identical to the two-sided transform of $x^{+}(n) = x(n)u(n)$. The ROC of its transform is always the exterior of a circle.

Examples:

$x_1(n) = \{1, 2, 5, 7, 0, 1\}$ $X_1^+(z) = 1 + 2z^{-1} + 5z^{-2} + 7z^{-3} + z^{-5}$

$x_2(n) = \{1, 2, \mathbf{5}, 7, 0, 1\}$ $X_2^+(z) = 5 + 7z^{-1} + z^{-3}$

$x_3(n) = \{0, 0, 1, 2, 5, 7, 0, 1\}$ $X_3^+(z) = z^{-2} + 2z^{-3} + 5z^{-4} + 7z^{-5} + z^{-7}$

$x_4(n) = \{2, 4, \mathbf{5}, 7, 0, 1\}$ $X_4^+(z) = 5 + 7z^{-1} + z^{-3}$

$x_5(n) = \delta(n)$ $X_5^+(z) = 1$

$x_6(n) = \delta(n-k)$ $X_6^+(z) = z^{-k}$

$x_7(n) = \delta(n+k)$ $X_7^+(z) = 0$

Note that $x_2(n) \neq x_4(n)$ but $X_2^+(z) = X_4^+(z)$ (obviously!).

The Shifting Property of the One-Sided Transform

Time Delay:

if:

$$x(n) \overset{z^+}{\longleftrightarrow} X^+(z)$$

then

$$x(n-k) \overset{z^+}{\longleftrightarrow} z^{-k}\left[X^+(z) + \sum_{n=1}^{k} x(-n)z^{n}\right] \qquad k > 0$$

for causal signals,

$$x(n-k) \overset{z^+}{\longleftrightarrow} z^{-k}X^+(z)$$

Proof:

$$Z^+\{x(n-k)\} = z^{-k}\left[\sum_{l=-k}^{-1} x(l)z^{-l} + \sum_{l=0}^{\infty} x(l)z^{-l}\right]$$

$$= z^{-k}\left[\sum_{l=-1}^{-k} x(l)z^{-l} + X^+(z)\right]$$

$$= z^{-k}\left[\sum_{n=1}^{k} x(-n)z^{n} + X^+(z)\right]$$

Time Advance:

$$x(n+k) \overset{z^+}{\longleftrightarrow} z^{k}\left[X^+(z) - \sum_{n=0}^{k-1} x(n)z^{-n}\right] \qquad k > 0$$

Example:

$$x(n) = a^n u(n)$$

$$X^+(z) = \frac{1}{1 - az^{-1}}$$

$$x(n) = a^{n-2}$$

$$\begin{aligned} X^+(z) &= z^{-2}[X^+(z) + x(-1)z^1 + x(-2)z^2] \\ &= z^{-2}X^+(z) + x(-1)z^{-1} + x(-2) \\ &= \frac{z^{-2}}{1 - az^{-1}} + a^{-1}z + a^{-2} \end{aligned}$$

Note the presence of only two terms from the $n < 0$ component.

Final Value Theorem

$$\lim_{n \to \infty} x(n) = \lim_{z \to 1} (z-1)X^{+}(z)$$

Example:

$$x(n) = u(n)$$

$$h(n) = a^{n}u(n)$$

What is the value of the output as $n \to \infty$?

$$Y(z) = \frac{1}{1-az^{-1}}\frac{1}{1-z^{-1}} = \frac{z^2}{(z-1)(z-a)}$$

Applying the final value theorem:

$$(z-1)Y(z) = \frac{z^2}{(z-a)}$$

$$\lim_{n \to \infty} y(n) = \lim_{z \to 1}\frac{z^2}{(z-a)} = \frac{1}{1-a}$$

Using the One-Sided Transform To Solve Difference Equations With Initial Conditions

Example:

$$y(n) = y(n-1) + y(n-2) \qquad y(-1) = 0, y(-2) = 1$$

$$\begin{aligned} Y^+(z) &= Z^+[y(n-1)] + Z^+[y(n-2)] \\ &= [z^{-1}Y^+(z) + y(-1)] + [z^{-2}Y^+(z) + y(-2) + y(-1)z^{-1}] \end{aligned}$$

or,

$$Y^+(z)[1 - z^{-1} - z^{-2}] = y(-1) + y(-2)$$

$$Y^+(z) = \frac{1}{1 - z^{-1} - z^{-2}}$$

from this, we can find the impulse response as:

$$y(n) = \frac{1}{\sqrt{5}}\left(\frac{1}{2}\right)^{n+1}[(1+\sqrt{5})^{n+1} - (1-\sqrt{5})^{n+1}]u(n)$$

Example:

$$x(n) = u(n)$$

$$y(n) = ay(n-1) + x(n) \qquad y(-1) = 1$$

$$Y^+(z) = a[z^{-1}Y^+(z) + y(-1)] + X^+(z)$$

or,

$$Y^+(z)[1 - az^{-1}] = a(1) + \frac{1}{1 - z^{-1}}$$

$$Y^+(z) = \frac{a}{1 - az^{-1}} + \frac{1}{(1 - z^{-1})(1 - az^{-1})}$$

therefore,

$$y(n) = \frac{1}{1-a}(1 - a^{n+2})u(n)$$