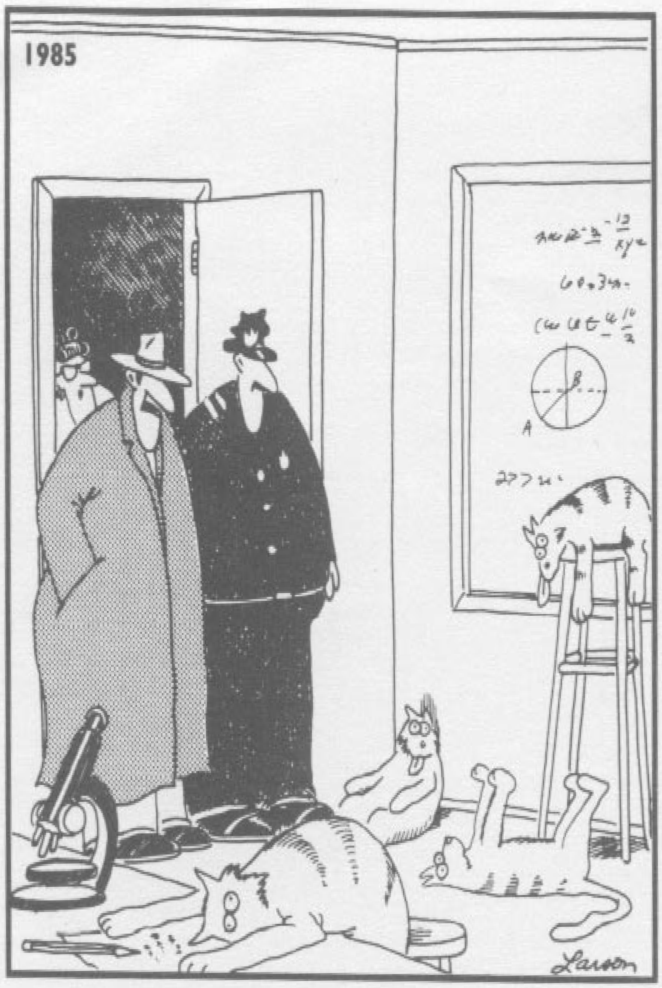


"Notice all the computations, theoretical scribblings, and lab equipment, Norm. … Yes, curiosity killed these cats."

The FarSide

September

28

THURSDAY

Poles and Zeroes

If $X(z)$ is a rational function,

$$X(z) = \frac{N(z)}{D(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots + b_M z^{-M}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_N z^{-N}} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

if $a_0 \neq 0$ and $b_0 \neq 0$, we can factor out $b_M z^{-M}$ and $a_N z^{-N}$:

$$X(z) = \frac{N(z)}{D(z)} = \left(\frac{b_0 z^{-M}}{a_0 z^{-N}}\right) \frac{z^M + \left(\frac{b_1}{b_0}\right) z^{M-1} + \left(\frac{b_2}{b_0}\right) z^{M-2} + \ldots + \left(\frac{b_M}{b_0}\right)}{z^N + \left(\frac{a_1}{a_0}\right) z^{N-1} + \left(\frac{a_2}{a_0}\right) z^{N-2} + \ldots + \left(\frac{a_N}{a_0}\right)}$$

$N(z)$ and $D(z)$ can be expressed in factored form:

$$X(z) = \frac{N(z)}{D(z)} = \left(\frac{b_0}{a_0} z^{-M+N}\right) \frac{(z - z_1)(z - z_2)\ldots(z - z_M)}{(z - p_1)(z - p_2)\ldots(z - p_N)}$$

or,

$$X(z) = (G z^{N-M}) \frac{\prod_{k=1}^{M} (z - z_k)}{\prod_{k=1}^{N} (z - p_k)}$$

where $G \equiv b_0 / a_0$.

A zero is defined as a value of z for which $X(z) = 0$.

A pole is defined as a value of z for which $X(z) \to \infty$.

$X(z)$ has M finite zeros at $\{z_M\}$.

$X(z)$ has N finite poles at $\{z_M\}$.

if $|N > M|$, $X(z)$ has $|N - M|$ zeros at $z = 0$ (repeated roots).

if $|N < M|$, $X(z)$ has $|N - M|$ poles at the origin (repeated roots).

Note also that, when $N \neq M$, $X(z)$ can have poles and zeros at $z = \infty$. In general, if we count the poles and zeros at $z = \infty$, $X(z)$ has the same number of poles and zeros.

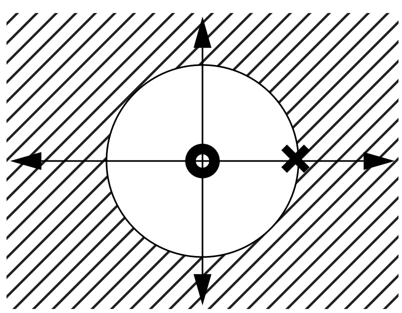
Obviously, the ROC of a z-transform should not contain any poles.

Example:

$$x(n) = a^n u(n)$$

$$X(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}, \qquad \text{ROC: } |z| > a$$

one zero at $z = 0$; one pole at $z = a$



Example:

$$x(n) = \begin{cases} a^n, & 0 \le n \le M-1 \\ 0, & elsewhere \end{cases}$$

Note: this is a finite duration signal!

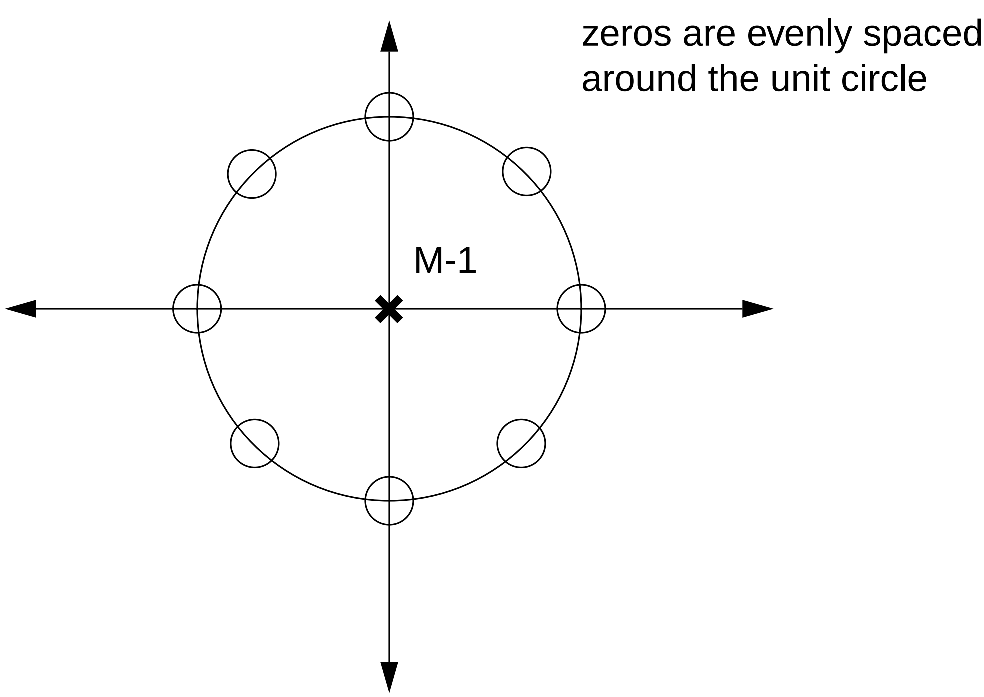
$$X(z) = \sum_{n=0}^{M-1} (az^{-1})^n = \frac{1-(az^{-1})^M}{1-az^{-1}} = \frac{z^M - a^M}{z^{M-1}(z-a)}$$

roots at $z^M = a^M$, which has M roots at

$$z_k = ae^{j2\pi\frac{k}{M}}, \qquad k = 0, 1, \ldots, M-1 .$$

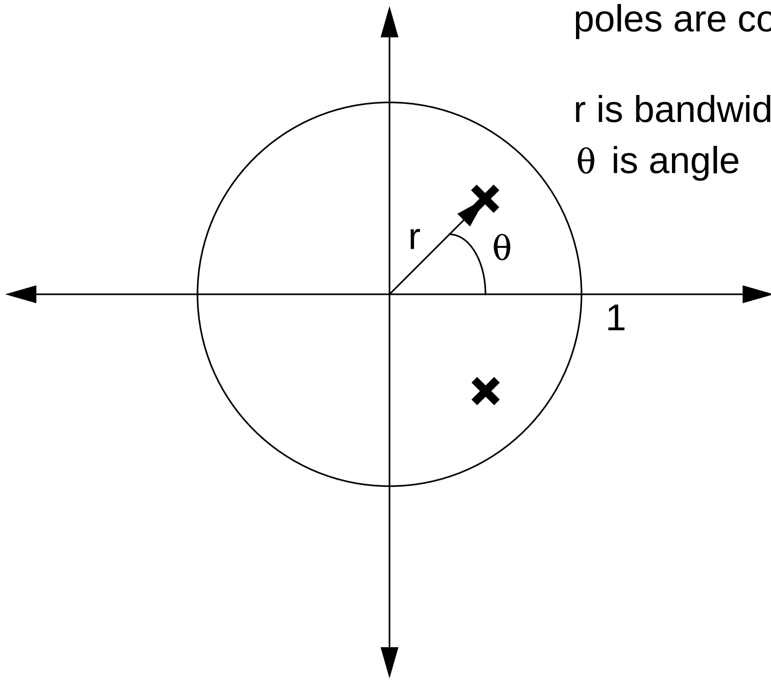
Note that the zero at $z = a$ cancels the corresponding pole.

Hence, finite duration signals are composed of ALL ZEROES!



Example:

$$
\begin{aligned}
X(z) &= \frac{1}{(1-az^{-1})(1-a^{*}z^{-1})} \\
&= \frac{1}{(1-(re^{j\theta})z^{-1})(1-(re^{-j\theta})z^{-1})} \\
&= \frac{1}{1-2r\cos\theta z^{-1}+r^{2}z^{-2}}
\end{aligned}
$$



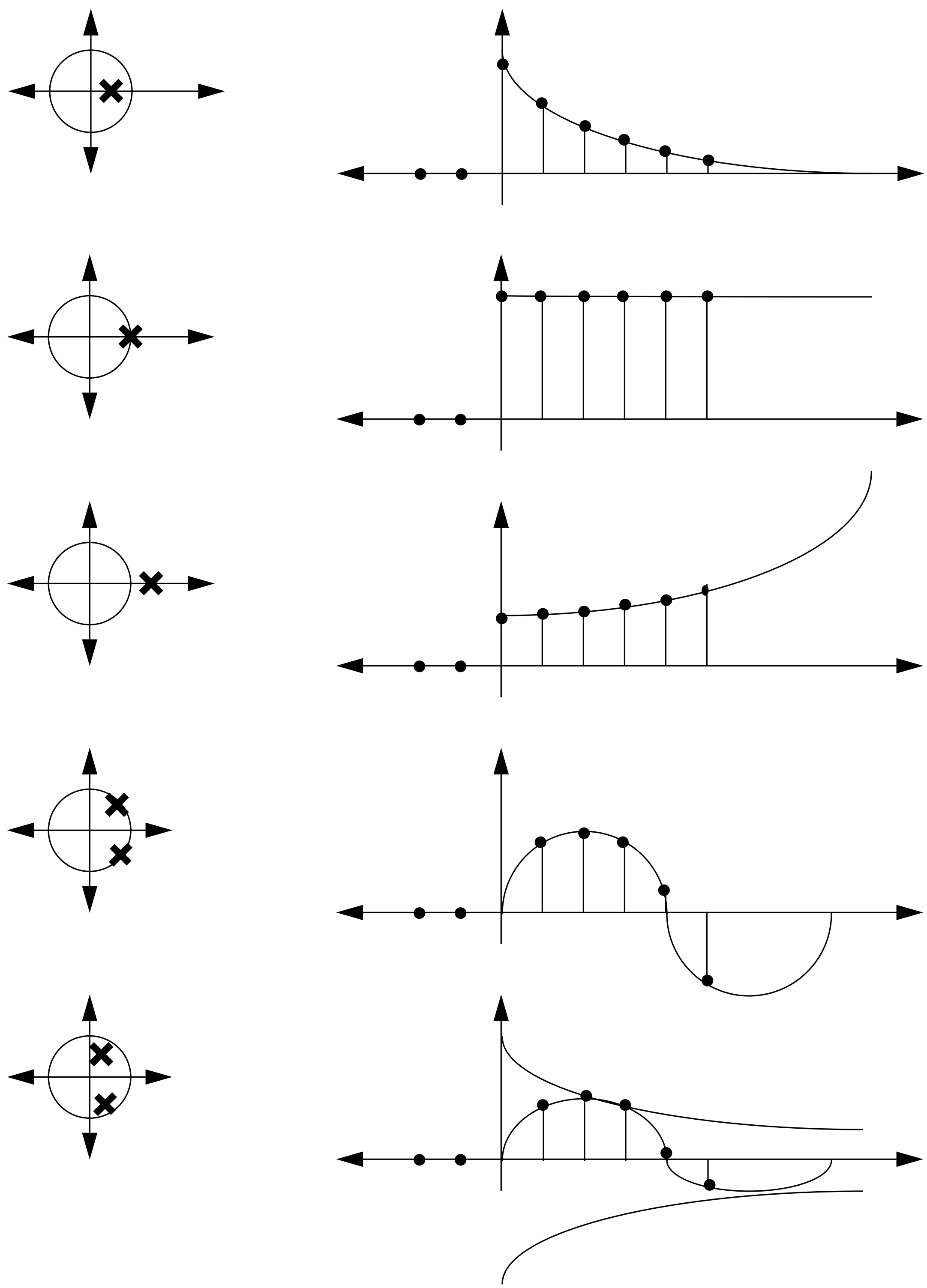
poles are complex conjugates:

r is bandwidth
θ is angle

NOTE THAT:

$|r| > 1$ implies instability
$\theta = \pi$ corresponds to $f = f_s / 2$

Pole Location and Time-Domain Behavior for Causal Signals



The System Function of a Linear Time-Invariant System

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

Take the z-transform:

$$Y(z) = -\sum_{k=1}^{N} a_k Y(z) z^{-k} + \sum_{k=0}^{M} b_k X(z) z^{-k}$$

$$Y(z) + \sum_{k=1}^{N} a_k Y(z) z^{-k} = \sum_{k=0}^{M} b_k X(z) z^{-k}$$

$$Y(z)\left(1 + \sum_{k=1}^{N} a_k z^{-k}\right) = X(z)\left(\sum_{k=0}^{M} b_k z^{-k}\right)$$

or,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

FIR Filter (all zeros/moving average): $H(z) = \sum_{k=0}^{M} b_k z^{-k}$

IIR All-Pole Filter (all poles/autoregressive): $H(z) = \frac{b_0}{1 + \sum_{k=1}^{N} a_k z^{-k}}$

The general case, FIR and IIR is obviously IIR, and is called an autoregressive moving average (ARMA) filter.

Example:

$$y(n) = \frac{1}{2}y(n-1) + 2x(n)$$

$$Y(z) = \frac{1}{2}z^{-1}Y(z) + 2X(z)$$

$$H(z) = \frac{2}{1 - \frac{1}{2}z^{-1}}$$

$$h(n) = 2\left(\frac{1}{2}\right)^{n}$$

Note:

$$H(\omega) = H(z)\big|_{z = e^{j\omega}} = \frac{2}{1 - \frac{1}{2}e^{-j\omega}}$$

Note: Think about implementing filters by decomposing the z-transform of the transfer function into components.