Properties of the z-Transform

Property	Time Domain	z-Domain
Notation	$x(n)$ $x_1(n)$ $x_2(n)$	$X(z)$ $X_1(z)$ $X_2(z)$
Linearity and Superposition	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(z) + a_2X_2(z)$
Time-Shifting	$x(n-k)$	$z^{-k}X(z)$
Scaling in the z-domain	$a^n x(n)$	$X(a^{-1}z)$
Time reversal	$x(-n)$	$X(z^{-1})$
Conjugation	$x^*(n)$	$X^*(z^*)$
Real part	$Re[x(n)]$	$\frac{1}{2}[X(z) + X^*(z^*)]$
Imag part	$Imag[x(n)]$	$\frac{1}{2j}[X(z) - X^*(z^*)]$
Differentiation in the z-domain	$nx(n)$	$-z\frac{d}{dz}X(z)$
Convolution	$x_1(n) \otimes x_2(n)$	$X_1(z)X_2(z)$
Correlation	$r_{x_1x_2}(l) = x_1(l) \otimes x_2(-l)$	$R_{x_1x_2}(z) = X_1(z)X_2(z^{-1})$

Properties of the z-Transform (cont.)

Property	Time Domain	z-Domain
Initial Value theor.	if $x(n)$ is causal	$x(0) = \lim_{z \to \infty} X(z)$
Multiplication	$x_1(n)x_2(n)$	$\frac{1}{2\pi j}\oint_C X_1(v)X_2(\frac{z}{v})v^{-1}dv$
Parseval's relation	$\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n) = \frac{1}{2\pi j}\oint_C X_1(v)X_2^*(\frac{1}{v^*})v^{-1}dv$	

Selected proofs:

(1) Linearity:

$$
\begin{aligned}
Z[a_1x_1(n) + a_2x_2(n)] &= \sum_{n=-\infty}^{\infty} (a_1x_1(n) + a_2x_2(n))z^{-n} \\
&= \sum_{n=-\infty}^{\infty} a_1x_1(n)z^{-n} + \sum_{n=-\infty}^{\infty} a_2x_2(n)z^{-n} \\
&= a_1\sum_{n=-\infty}^{\infty} x_1(n)z^{-n} + a_2\sum_{n=-\infty}^{\infty} x_2(n)z^{-n} \\
&= a_1X_1(z) + a_2X_2(z)
\end{aligned}
$$

(2) Convolution:

$$x(n) = \sum_{k=-\infty}^{\infty} x_1(k)x_2(n-k)$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} = \sum_{n=-\infty}^{\infty}\left[\sum_{k=-\infty}^{\infty} x_1(k)x_2(n-k)\right]z^{-n}$$

Interchange the order of summation:

$$= \sum_{k=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} x_1(k)x_2(n-k)z^{-n}$$

$$= \sum_{k=-\infty}^{\infty} x_1(k)\sum_{n=-\infty}^{\infty} x_2(n-k)z^{-n}$$

Using the time-shifting property:

$$= \sum_{k=-\infty}^{\infty} x_1(k)z^{-k}X_2(z)$$

$$= X_2(z)\sum_{k=-\infty}^{\infty} x_1(k)z^{-k}$$

$$= X_2(z)X_1(z) = X_1(z)X_2(z)$$

Example:

$$x(n) = [3(2^n) - 4(3^n)]u(n)$$

$$X(z) = 3Z[2^n] - 4Z[3^n]$$

or,

$$X(z) = \frac{3}{1-2z^{-1}} - \frac{4}{1-3z^{-1}}$$

The region of convergence for the second term is a subset of the region of convergence for the first term. Hence,

$$\text{ROC:} \quad |z| > 3$$

Example:

$$x(n) = \cos(\omega_0 n)u(n)$$

From Euler's identity:

$$x(n) = \frac{1}{2}e^{j\omega_0 n}u(n) + \frac{1}{2}e^{-j\omega_0 n}u(n)$$

$$X(z) = \frac{1}{2}\frac{1}{1-e^{j\omega_0}z^{-1}} + \frac{1}{2}\frac{1}{1-e^{-j\omega_0}z^{-1}}$$

Example:

$$x(n) = a^n\cos(\omega_0 n)u(n)$$

$$X(z) = X(a^{-1}z)$$

or,

$$X(z) = \frac{1}{2}\frac{1}{1-ae^{j\omega_0}z^{-1}} + \frac{1}{2}\frac{1}{1-ae^{-j\omega_0}z^{-1}}$$

Example:

$$x_1(n) = \{1, -2, 1\}$$
$$x_2(n) = \{1, 1, 1, 1, 1, 1\} = u(n) - u(n-6)$$

$$X_1(z) = 1 - 2z^{-1} + z^{-2}$$
$$X_2(z) = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5}$$

therefore,

$$\begin{aligned} Y(z) &= X_1(z)X_2(z) \\ &= 1 - z^{-1} - z^{-6} + z^{-7} \\ x(n) &= \{1, -1, 0, 0, 0, 0, -1, 1\} \end{aligned}$$

Example:

$$x(n) = a^n u(n)$$

Compute the autocorrelation function:

$$\begin{aligned} R_{xx}(z) &= X(z)X(z^{-1}) \\ &= \frac{1}{1 - az^{-1}} \frac{1}{1 - az} \end{aligned}$$

$$r_{xx}(l) = \frac{1}{1 - a^2} a^{|l|}$$

How do we find the energy density spectrum?