Resolution of a Discrete-Time Signal Into Impulses:

$$x(n) = \sum_k c_k x_k(n)$$

Suppose:

$$x_k(n) = \delta(n-k)$$

then:

$$x(n)\delta(n-k) = x(k)\delta(n-k)$$

is zero everywhere except at $n = k$.

This means we can write $x(n)$ as:

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k)$$

Example:

$$x(n) = \{2, \mathbf{4}, 0, 3\}$$

$$x(n) = 2\delta(n+1) + 4\delta(n) + 3\delta(n-2)$$

Response of LTI Systems to Arbitrary Inputs: Discrete Convolution

$$
\begin{aligned}
y(n) &= H[x(n)] \\
&= H\left[\sum_{k=-\infty}^{\infty} x(k)\delta(n-k)\right] \\
&= \sum_{k=-\infty}^{\infty} x(k)H[\delta(n-k)]
\end{aligned}
$$

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

$$y(n) \equiv x(n) \otimes h(n)$$

This is discrete convolution, analogous to convolution in analog systems.

Steps:

(1) Folding. Fold $h(k)$ about $k = 0$ to obtain $h(-k)$.

(2) Shifting. Shift $h(-k)$ to obtain $h(n-k)$.

(3) Multiply sequences.

(4) Sum.

Example (see book):

$$h(n) = \{1, \mathbf{2}, 1, -1\}$$

$$x(n) = \{\mathbf{1}, 2, 3, 1\}$$

$$y(n) = \{\ldots, 0, 0, 1, 4, 8, 8, 3, -2, -1, 0, 0, \ldots\}$$

Discrete convolution is commutative:

$$y(n) = \sum_{k=-\infty}^{\infty} x(n-k)h(k)$$

Example:

$$x(n) = u(n)$$

$$h(n) = a^n u(n)$$

$$y(0) = 1$$
$$y(1) = 1 + a$$
$$y(2) = 1 + a + a^2$$
$$y(n) = \frac{1-a^{n+1}}{1-a}$$

Simple Properties:

Commutative Law:

$$x(n) \otimes h(n) = h(n) \otimes x(n)$$

Associative Law:

$$(x(n) \otimes h_1(n)) \otimes h_2(n) = x(n) \otimes [h_1(n) \otimes h_2(n)]$$

Distributive Law:

$$x(n) \otimes [h_1(n) + h_2(n)] = x(n) \otimes h_1(n) + x(n) \otimes h_2(n)$$

Causality:

> An LTI system is causal if and only if its impulse response is zero for negative values of n.

This is obvious when you recall the folding step in discrete convolution.

If,

$$h(n) = 0 \qquad n < 0$$

Then,

$$y(n) = \sum_{k=0}^{n} x(k)h(n-k)$$

Example:

$$x(n) = u(n)$$

$$h(n) = a^n u(n)$$

By direction summation using the above equation:

$$y(n) = \frac{1-a^{n+1}}{1-a}$$

Stability of Linear Time-Invariant Systems:

A linear system is stable if its impulse response is absolutely summable (necessary and sufficient condition):

$$S_h \equiv \sum_{k=-\infty}^{\infty} |h(k)| < \infty$$

implies the impulse response will go to zero as N approaches infinity

This means that if the input is bounded, and the system is stable, the output will be bounded:

Bounded **I**nput / **B**ounded **O**utput (BIBO)

NOTE: A very useful thing to remember when debugging programs!

Example: Is $h(n)$ stable?

$$h(n) = a^n u(n)$$

Compute sum:

$$\sum_{k=0}^{\infty} \left|a^k\right| = \sum_{k=0}^{\infty} |a|^k = 1 + |a| + |a|^2 + \dots$$

From a mathematical handbook, this geometric series converges:

$$\sum_{k=0}^{\infty} |a|^k = \frac{1}{1-|a|}$$

Classification of Systems:

Finite Impulse Response (FIR):

for causal, FIR systems:

$$h(n) = 0 \qquad n < 0 \text{ and } n \geq M$$

convolution reduces to:

$$y(n) = \sum_{k=0}^{M-1} h(k)x(n-k)$$

Infinite Impulse Response (IIR):

$$y(n) = \sum_{k=0}^{\infty} h(k)x(n-k)$$