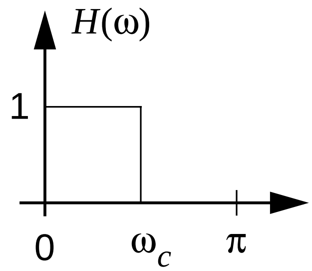
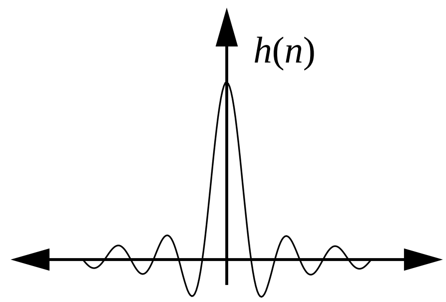
Causality and Its Implications



Is it possible to realize an ideal filter in practice?

$$H(\omega) = \begin{cases} 1, & |\omega| \le \omega_c \\ 0, & \omega_c < |\omega| \le \pi \end{cases}$$

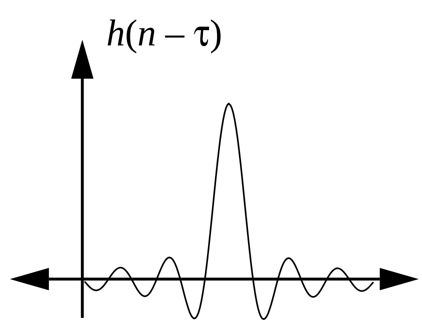
The impulse response of this filter is:



$$h(n) = \begin{cases} \dfrac{\omega_c}{\pi}, & |\omega| \le \omega_c \\ \dfrac{\omega_c}{\pi}\dfrac{\sin \omega_c n}{\omega_c n}, & \omega_c < |\omega| \le \pi \end{cases}$$

It is clear that the ideal lowpass filter:

- is noncausal
- is unrealizable
- has an impulse response that is not absolutely summable
- is unstable
- has a main lobe whose width is inversely proportional to the bandwidth



What happens if we truncate the filter and delay (shift) the impulse response?

Paley-Weiner Theorem: If $h(n)$ has finite energy and $h(n) = 0$ for $n < 0$, then

$$\int_{-\pi}^{\pi} |(\log|H(\omega)|)d\omega| < \infty$$

Conversely, if $|H(\omega)|$ is square integrable and if the above integral is finite, then we can associate a phase response $\Theta(\omega)$ so that the resulting filter with frequency response $H(\omega) = |H(\omega)|e^{j\Theta(\omega)}$ is causal.

Note: This implies $|H(\omega)|$ can be zero at some points, but not zero over some finite interval.

Relationships Between Real and Imaginary Components of the Fourier Transform for Causal Signals

Recall that $h(n)$ can be decomposed into a real and imaginary part:

$$h(n) = h_e(n) + h_o(n)$$

where

$$h_e(n) = \frac{1}{2}[h(n) + h(-n)] \qquad h_o(n) = \frac{1}{2}[h(n) - h(-n)]$$

If $h(n)$ is causal, it is possible to recover $h(n)$ from its even part $h_e(n)$ for $0 \le n \le \infty$ or from its odd part $h_o(n)$ for $1 \le n \le \infty$. From the above equations:

$$h(n) = 2h_e(n)u(n) - h_e(0)\delta(n) \qquad n \ge 0$$

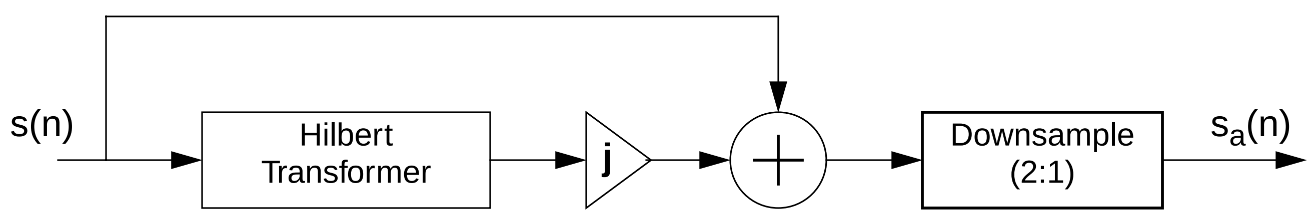
$$h(n) = 2h_o(n)u(n) + h(0)\delta(n) \qquad n \ge 1$$

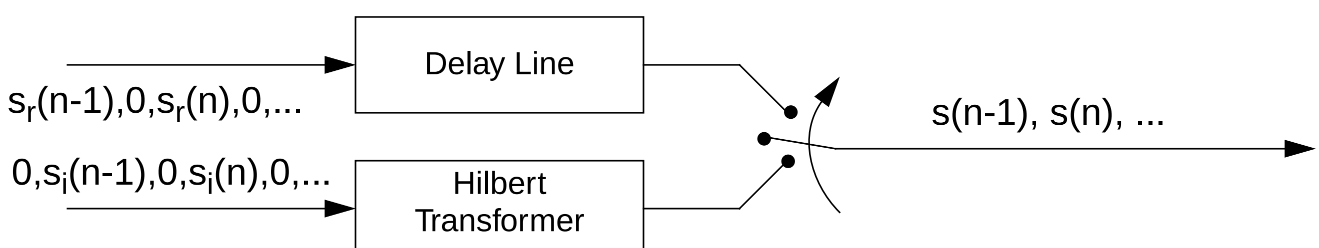
Note that $h_e(n) = h_o(n)$ for $n > 0$, and that to recover $h(n)$ from $h_o(n)$, we must know $h(0)$.

By taking the Fourier transform of the above expression for $h_e(n)$, we can show the relationship between $H_R(\omega)$ and $H_I(\omega)$:

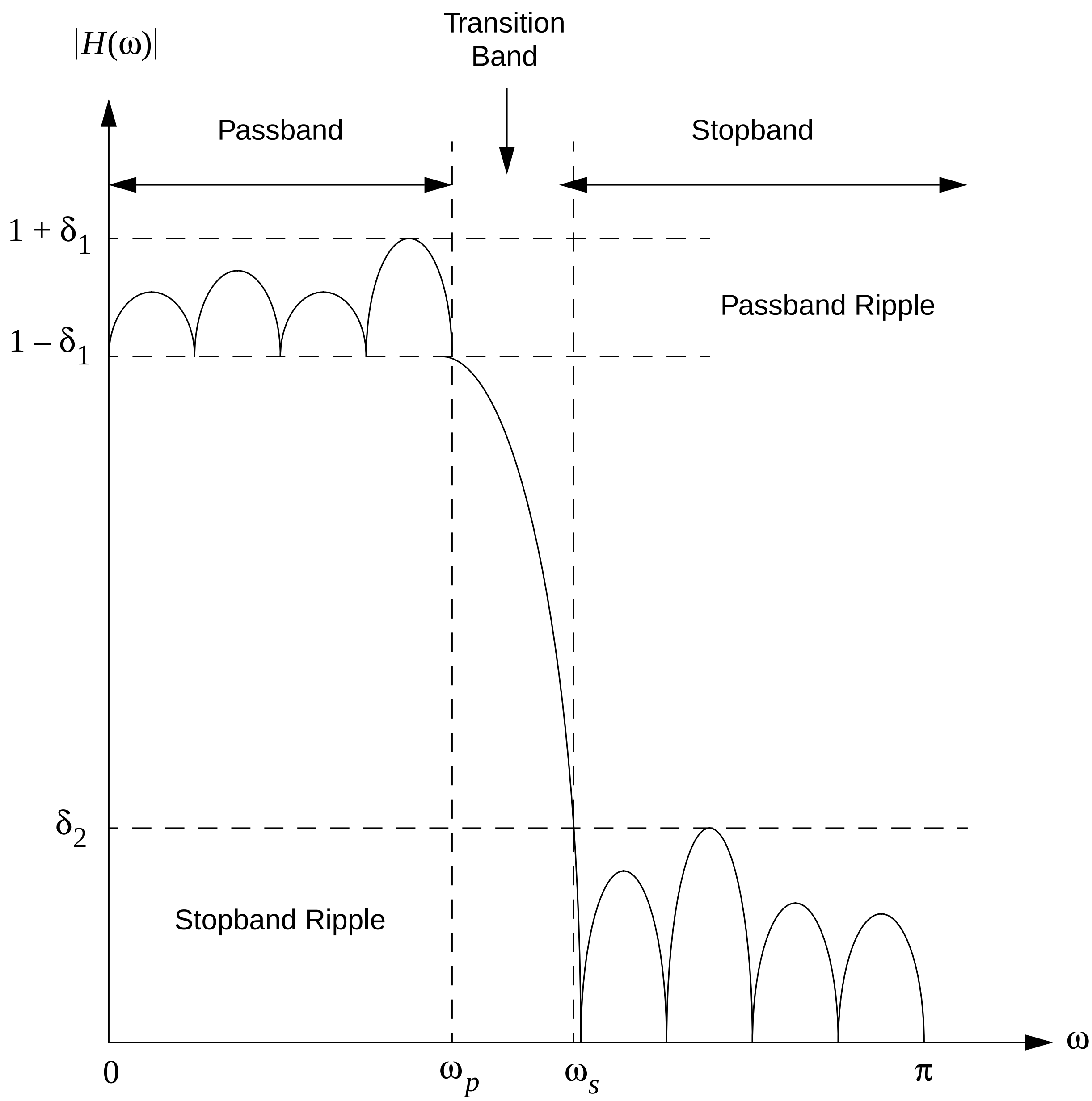
$$H_I(\omega) = -\frac{1}{2\pi}\int_{-\pi}^{\pi} H_R(\omega)\cot\frac{\omega-\lambda}{2}d\lambda$$

This integral is called a discrete Hilbert transform. Lest you think this is some unrealizable mathematical abstraction, this operation can be implemented with the system shown below:





The Nomenclature of Digital Filters



Note: The best trade-off of these parameters is most often highly application dependent!

The Design of Linear-Phase Filters (A Frequency Sampling Approach)

An FIR filter of length M has a frequency response:

$$H(z) = \sum_{k=0}^{M-1} b_k z^k \qquad\qquad H(\omega) = \sum_{k=0}^{M-1} b_k e^{-j\omega k}$$

where the filter coefficients are also samples of the impulse response:

$$h(n) = \begin{cases} b_n, & 0 \le n < M-1 \\ 0, & otherwise \end{cases}$$

Consider the case where:

$$h(n) = h(M-1-n)$$

It is straightforward to show that the frequency response of such a filter is given by:

$$H(\omega) = H_r(\omega) e^{-j\omega(M-1)/2}$$

where

$$H_r(\omega) = h(\frac{M-1}{2}) + 2\sum_{n=0}^{(M-3)/2} h(n)\cos\omega\left(\frac{M-1}{2} - n\right), \qquad M\ odd$$

$$H_r(\omega) = 2\sum_{n=0}^{(M/2)-1} h(n)\cos\omega\left(\frac{M-1}{2} - n\right), \qquad M\ even$$

The phase characteristics for both filters is a simple delay:

$$\angle H_r(\omega) = \begin{cases} -\omega\left(\dfrac{M-1}{2}\right), & H_r(\omega) \ge 0 \\ -\omega\left(\dfrac{M-1}{2}\right) + \pi & H_r(\omega) < 0 \end{cases}$$

It is also possible to design linear phase filters with the constraint:

$$h(n) = -h(M-1-n)$$

Recall this is an antisymmetric linear phase filter.

The previous equations define a system of linear equations that specify samples of the impulse response in terms of samples of the frequency response.

If we select uniformly spaced samples in frequency:

$$\omega_k = 2\pi\left(\frac{k}{M}\right), \qquad \begin{array}{ll} k = 0, 1, \ldots, \dfrac{M-1}{2} & M \; odd \\ k = 0, 1, \ldots, \dfrac{M}{2}-1 & M \; even \end{array}$$

we can write the following equations:

$$\sum_{n=0}^{(M-1)/2} a_{kn} h(n) = H_r(\omega_k) \qquad k = 0, 1, \ldots, \frac{M-1}{2} \qquad M \; odd$$

$$\sum_{n=0}^{\left(\frac{M}{2}\right)-1} a_{kn} h(n) = H_r(\omega_k) \qquad k = 0, 1, \ldots, \frac{M}{2}-1 \qquad M \; even$$

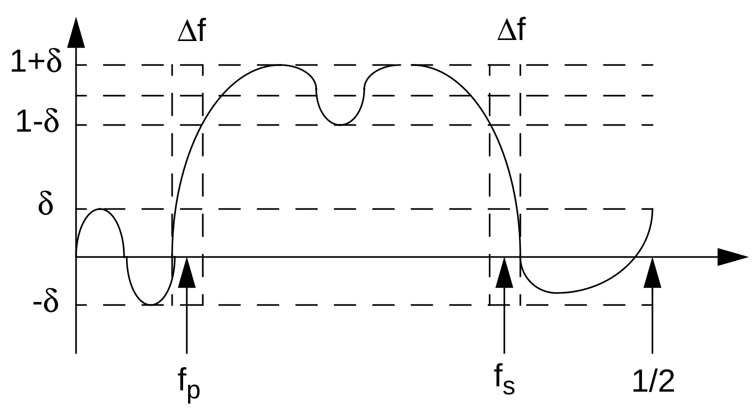
where

$$a_{kn} = 2\cos\left[\omega_k\left(\frac{M-1}{2} - n\right)\right] \qquad n \neq \frac{M-1}{2}$$

$$a_{kn} = 1, \qquad \left(n = \frac{M-1}{2}\right), \; all \; k$$

These equations can be solved using a standard linear equation solver.

Alternate design equations are available if we constrain the shape of the impulse response in the time domain. One important algorithm is the Kaiser window design:



The Kaiser Window Filter Design
(Good Housekeeping Seal of Approval!)

Steps:

1. Compute the attenuation:

$$A = -20\log_{10}\delta$$

2. Compute the filter order (ΔF is the bandwidth):

$$N = \frac{A - 7.95}{28.72\Delta F} \text{ (round upwards)}$$

If N is acceptable:

3. $$\alpha = \begin{cases} 0.1102(A - 8.7) & 50 \le A \\ 0.5842(A - 21)^{0.4} + 0.07886(A - 21) & 21 < A < 50 \\ 0 & A \le 21 \end{cases}$$

4. $$c_0 = 2(f_s - f_p)$$
$$c_k = \frac{1}{\pi k}[\sin 2\pi k f_s - \sin 2\pi k f_p] \qquad (k = 1, 2, \ldots, N)$$

5. Compute $I_0(\alpha)$ using (can be computed recursively):

$$I_0(x) = 1 + \sum_{n=1}^{\infty} \left[\frac{(x/2)^n}{n!}\right]$$

6. Compute the window weights:

$$w_k = \begin{cases} \dfrac{I_0\left[\alpha\sqrt{1 - (k/N)^2}\right]}{I_0(\alpha)} & |k| \le N \\ 0 & |k| > N \end{cases}$$

7. The final filter coefficients are:

$$h(n) = c_n w_n$$

The resulting frequency response is:

$$\tilde{H}(f) = c_0 + 2\sum_{k=1}^{N} c_k w_k \cos(2\pi k f) \qquad \left(0 \le f \le \frac{1}{2}\right)$$