Efficient Computation of the Discrete Fourier Transform (DFT)

Recall the DFT:

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}, \qquad k = 0, 1, 2, \ldots, N-1$$

or,

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}, \qquad k = 0, 1, 2, \ldots, N-1$$

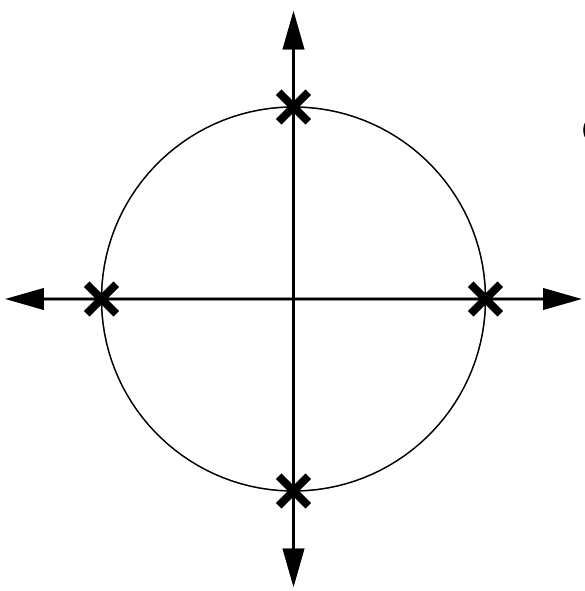
where $W_N = e^{-j2\pi/N}$ and $W_N^{kn} = e^{-j2\pi kn/N}$.

Note that W_N^{kn} are just samples on the unit circle:

N = 4

k = 0, 1, 2, 3

n = 0, 1, 2, 3



Only four unique values!

For example, $W_4^{(3)(2)} = e^{(-j2\pi/4)(3)(2)} = e^{-j3\pi} = e^{-j\pi} = -1$.

We note two important symmetry properties of W_N^{kn}:

$$W_N^{k+N/2} = -W_N^k \qquad \text{(symmetry about the imaginary axis)}$$

$$W_N^{k+N} = W_N^k \qquad \text{(periodicity)}$$

This symmetry allows the number of computations for a DFT to be reduced significantly.

Computational Complexity

For a complex-valued sequence:

$$X_R(k) = \sum_{n=0}^{N-1}\left[x_R(n)\cos\left(\frac{2\pi kn}{N}\right) + x_I(n)\sin\left(\frac{2\pi kn}{N}\right)\right]$$

$$X_I(k) = -\sum_{n=0}^{N-1}\left[x_R(n)\sin\left(\frac{2\pi kn}{N}\right) - x_I(n)\cos\left(\frac{2\pi kn}{N}\right)\right]$$

Direct computation requires:

1. $2N^2$ evaluations of trig functions (typically performed using table lookup — a trade-off of memory for speed)
2. $4N^2$ real multiplications
3. $4N(N-1)$ real additions
4. Misc. indexing and addressing operations

In general, we say that the complexity is $O(N^2)$ — which implies it is not linearly proportional to the length of the input.

Why is this bad?

Divide and Conquer

Consider the case $N = LM$ (N can be factored into a product of two integers):

n=0	n=1	n=2	• • •	n=N-1
$x(0)$	$x(1)$	$x(2)$	$\ldots$	$x(N-1)$

Consider the mapping: $n = l + mL$:

l/m	0	1	• • •	M-1
0	$x(0)$	$x(L)$	$\ldots$	$x((M-1)L)$
1	$x(1)$	$x(L+1)$	$\ldots$	$x((M-1)L+1)$
2	$x(2)$	$x(L+2)$	$\ldots$	$x((M-1)L+2)$
• • •	$\ldots$	$\ldots$	$\ldots$	$\ldots$
L-1	$x(L-1)$	$x(2L-1)$	$\ldots$	$x(ML-1)$

We can similarly map the DFT index k using $k = Mp + q$ (or $k = qL + p$).

The DFT can be computed as:

$$X(p, q) = \sum_{l=0}^{L-1} \left\{ W_N^{lq} \left[\sum_{m=0}^{M-1} x(l, m) W_M^{mq} \right] \right\} W_L^{lp}$$

The inner term represents an M-point DFT, while the outer term represents an L-point DFT. What is the advantage of this approach?

Example: N=1000

Normal DFT (complexity N^2): 10^6 operations

Divide and Conquer ($L = 2,\ M = 500$): $5x10^5$ operations (2x reduction)

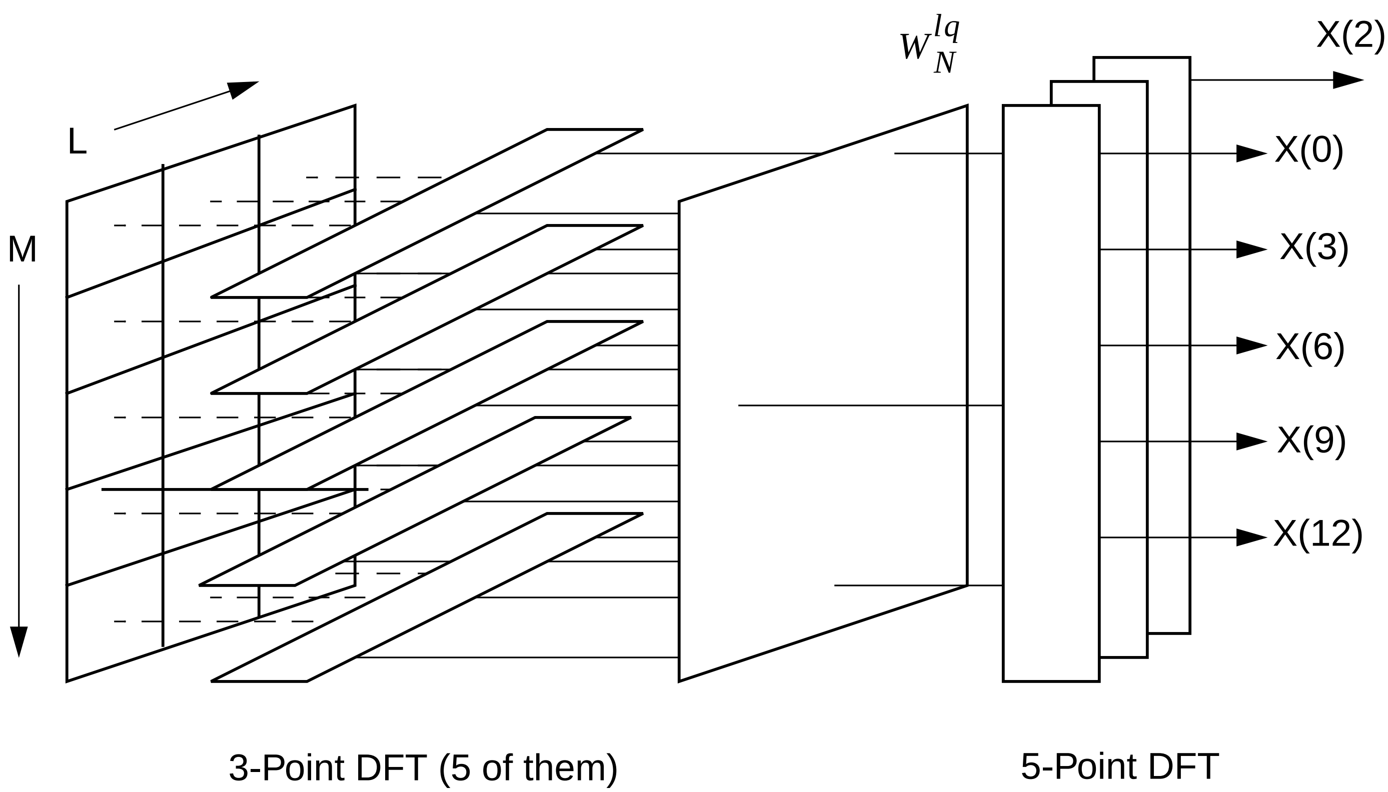
In general, the complexity of the divide and conquer approach is:

$N(M + L + 1)$ complex multiplications
$N(M + L - 2)$ additions

The complexity is reduced from $O(N^2)$ to something less...

Example: Computation of a 15-Point DFT

An example of the flow of computations for a 15-point DFT decomposed into $L = 3$ and $M = 5$:



For N highly composite (can be factored into a product of small prime numbers):

$$N = r_1 r_2 r_3 \ldots r_v$$

we can decompose a DFT of large order into a sequence of small DFTs.

For example,

$$N = 210 = 2x3x5x7$$

Suppose N is prime?

Can we impose additional constraints to further improve efficiency?