Frequency-Domain Sampling

Recall the Fourier transform of an aperiodic discrete-time signal:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

Evaluate this at equidistant samples $\omega = \frac{2\pi}{N}k$:

$$X\left(\frac{2\pi}{N}k\right) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\frac{2\pi}{N}kn}, \qquad k = 0, 1, 2, \ldots, N-1$$

The summation can be subdivided:

$$X\left(\frac{2\pi}{N}k\right) = \ldots + \sum_{n=-N}^{-1} x(n)e^{-j2\pi kn/N} + \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}$$

$$+ \sum_{n=N}^{2N-1} x(n)e^{-j2\pi kn/N} + \ldots$$

$$= \sum_{l=-\infty}^{\infty} \sum_{n=lN}^{lN+N-1} x(n)e^{-j2\pi kn/N} \qquad k = 0, 1, 2, \ldots, N-1$$

If we interchange the summations:

$$X\left(\frac{2\pi}{N}k\right) = \sum_{n=0}^{N-1}\left[\sum_{l=-\infty}^{\infty} x(n-lN)\right]e^{-j2\pi kn/N}$$

The signal

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n-lN)$$

is obtained by the periodic repetition of $x(n)$ every N samples. Its Fourier Series is given by:

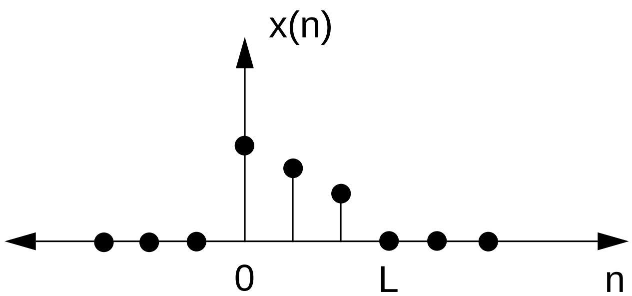
$$x_p(n) = \sum_{n=0}^{N-1} c_k e^{j2\pi kn/N} \qquad \text{and} \qquad c_k = \frac{1}{N}\sum_{n=0}^{N-1} x_p(n)e^{-j2\pi kn/N}$$

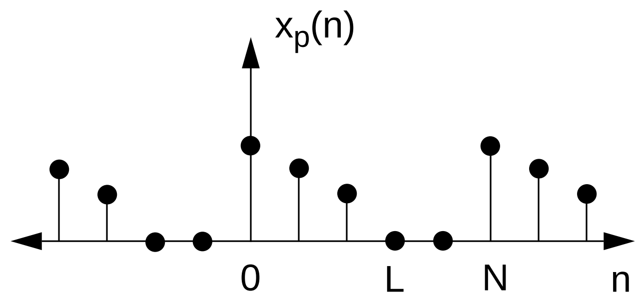
Reconstruction from the Fourier Transform

Since $x_p(n)$ is the periodic reconstruction of $x(n)$,

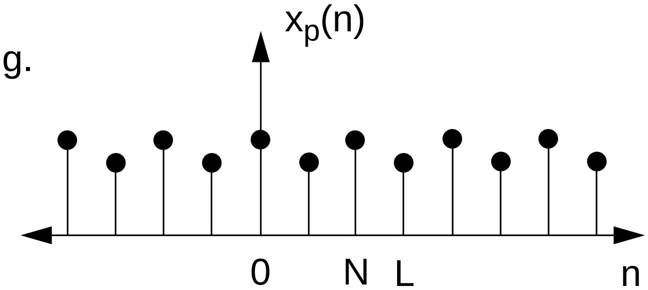
$$x(n) = x_p(n), \qquad 0 \le n \le N-1$$

ONLY when $N \ge L$:





If $N \le L$, we have time-domain aliasing.



Assuming $N \ge L$,

$$x(n) = \sum_{k=0}^{N-1} X\left(\frac{2\pi}{N}k\right) e^{j2\pi kn/N}$$

We can write the Fourier transform in terms of $X\left(\frac{2\pi}{N}k\right)$:

$$X(\omega) = \sum_{n=0}^{N-1} \left[\frac{1}{N} \sum_{k=0}^{N-1} X\left(\frac{2\pi}{N}k\right) e^{j2\pi kn/N} \right] e^{-j\omega n}$$

This can be simplified to:

$$X(\omega) = \sum_{k=0}^{N-1} X\left(\frac{2\pi}{N}k\right) P\left(\omega - \frac{2\pi}{N}k\right), \qquad N \ge L$$

where

$$P(\omega) = \frac{\sin(\omega N/2)}{N \sin(\omega/2)} e^{-j\omega(N-1)/2}$$

What is the significance of this equation?

Frequency Domain Interpolation Via Zero-Padding

Suppose $x(n) = 0$ for $n < 0$ and $n \geq L$. Let us define $x_p(n)$:

$$x_p(n) = \begin{cases} x(n), & 0 \leq n \leq L-1 \\ 0, & L \leq n \leq N-1 \end{cases}$$

We define the discrete Fourier transform (DFT) of $x(n)$ as:

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}, \qquad k = 0, 1, 2, \ldots, N-1$$

and the inverse discrete Fourier transform (IDFT):

$$x(n) = \sum_{n=0}^{N-1} X(k) e^{j2\pi kn/N}, \qquad n = 0, 1, 2, \ldots, N-1$$

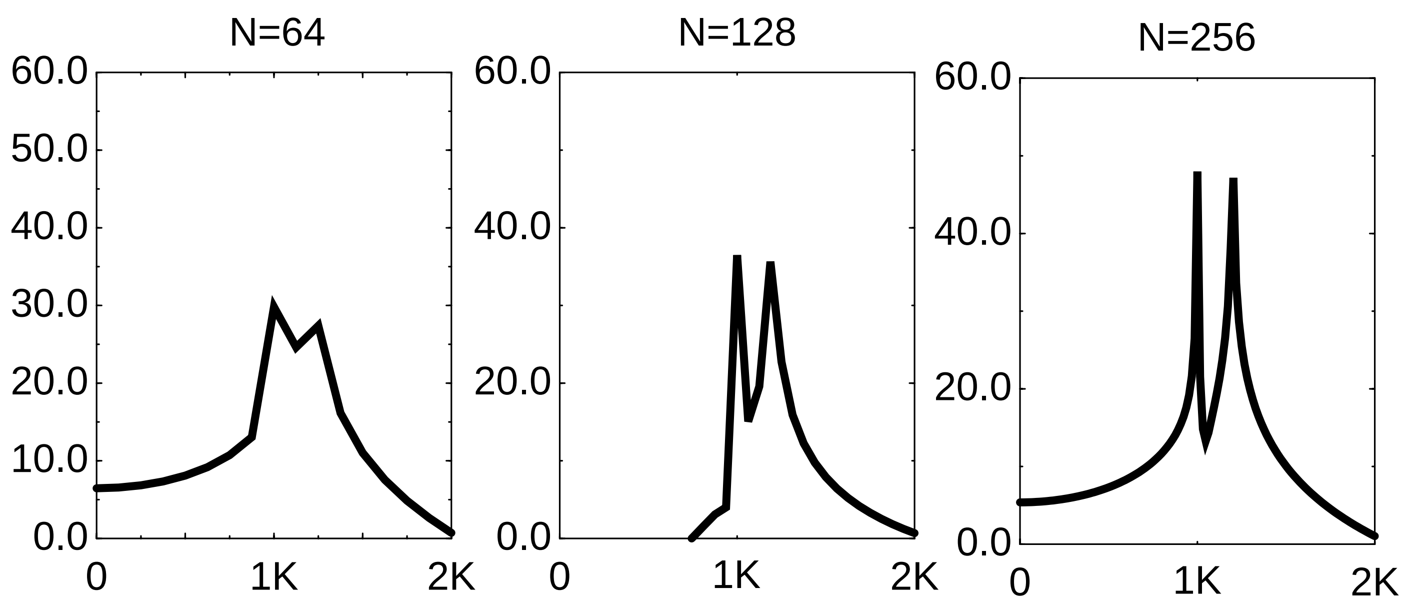
Clearly, if $L < N$, $x(n) = 0$ for $L \leq n \leq N-1$.

Important questions:

What is the meaning of life?

Why $N \gg L$?

Example: f_1 = 999.0 Hz, f_2 = 1199.0 Hz, f_s = 8000.0 Hz



The DFT as a Linear Transformation

We can rewrite the DFT as follows:

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \qquad k = 0, 1, \ldots, N-1$$

$$x(n) = \sum_{n=0}^{N-1} X(k) W_N^{-kn} \qquad n = 0, 1, \ldots, N-1$$

where $W_N = e^{-j2\pi/N}$, which is an Nth root of unity.

We can express the DFT as a matrix operation by defining:

$$\boldsymbol{x}_N = \begin{bmatrix} x(0) \\ x(1) \\ \ldots \\ x(N-1) \end{bmatrix} \qquad \boldsymbol{X}_N = \begin{bmatrix} X(0) \\ X(1) \\ \ldots \\ X(N-1) \end{bmatrix}$$

$$\boldsymbol{W}_N = \begin{bmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & W_N^1 & W_N^2 & \ldots & W_N^{N-1} \\ 1 & W_N^2 & W_N^4 & \ldots & W_N^{2(N-1)} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & \ldots & W_N^{(N-1)(N-1)} \end{bmatrix}$$

and rewriting the DFT as:

$$\boldsymbol{X}_N = \boldsymbol{W}_N \boldsymbol{x}_N$$

and the inverse DFT as:

$$\boldsymbol{x}_N = \boldsymbol{W}_N^{-1} \boldsymbol{X}_N$$

DFT Matrix Operations

Note that, given our definition of $\boldsymbol{W}_N$, the IDFT can be expressed as:

$$\boldsymbol{x}_N = \frac{1}{N}\boldsymbol{W}^*{}_N\boldsymbol{X}_N$$

Therefore, we can equate IDFT equations and write:

$$\boldsymbol{W}_N^{-1} = \frac{1}{N}\boldsymbol{W}^*{}_N$$

which, in turn, implies that

$$\boldsymbol{W}_N\boldsymbol{W}^*{}_N = N\boldsymbol{I}_N$$

where $\boldsymbol{I}_N$ is an NxN identity matrix.

Therefore, the matrix $\boldsymbol{W}_N$ is an orthogonal (unitary) matrix, and its inverse exists.

More important questions:

If a tree falls in the forest...

Is $\boldsymbol{W}_N$ redundant?

Can we simply the number of computations required for $\boldsymbol{X}_N$?